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Bosonic String Theory and T-duality

Florian Langer

supervised by
Prof. Daniel Waldram
assessed by
Prof. Kellogg Stelle

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To my mom who loves me,
my dad who believes in me
and my sister who inspires me.

Abstract

As string theory may serve as both, a theory of quantum gravity and a unified theory of all fundamental forces, it has naturally been one of the most active research fields of theoretical physics in the last decades. Known for its notorious difficulty, this report aims to be a simple introduction to bosonic string theory and one of its beautiful symmetries, T-duality. Unlike many other textbooks or lecture notes, this report explicitly includes all of the calculation it relies on as appendices. Hopefully this will help other students like me on their entry to the world of strings.

Starting off with the general idea and a brief historic background, the classical string is analysed. The Polyakov action is investigated in terms of its equations of motion and symmetries. These symmetries are used to fix the flat gauge in which the equations of motion take a particularly simple form. Following a naive quantisation procedure the constraints arising from the equations of motion are imposed as operator conditions on the Hilbert space. This Hilbert space contains negative norm states, but using the residual conformal symmetry it is shown how these can be removed. In a second method of quantisation, the BRST procedure, the original symmetry is replaced by a larger symmetry that is still present after gauge fixing. It is demonstrated that physical states appear as the cohomology of the generator of the BRST symmetry. Finally the concept of compactification is introduced and it is shown how string theories with differing compactified dimensions are equivalent under T-duality.

Contents

1 Introduction

1.1 The Standard Model

Our current understanding is that there are four forces at play in our universe. Three of these forces, the electromagnetic, the weak and the strong force are described extremely well by the Standard Model (SM). Even though the SM is one of the greatest scientific achievements of the 20th century it has two major shortcomings: First, it has approximately twenty parameters that have to be entered into the framework manually. While it is surely desirable to have a predictive theory that does not rely on so many free parameters, the second, and more severe, problem is that the SM does not include the fourth force, gravity. However, unlike the SM, which is set in a quantum framework, gravity is described by the classical theory of General Relativity. Any straightforward attempt to quantise gravity such that it can be incorporated into the SM faces great difficulties due to divergences in the ultraviolet. So far there are only very few consistent theories describing quantum gravity, string theory being one of them.

1.2 String Theory

String theory is a bold venture. It not only describes quantum gravity, but aspires to be an all-encompassing theory of our universe. Its basic assumption is that our universe is not built from point-like particles, but from 1-dimensional strings. It postulates that what so far has been perceived as elementary particles are just different vibrational modes of one fundamental string. Among the vibrational modes of the string is one that looks like the graviton. This is amazing as at now point in the theory is gravity put in manually. It naturally arises as the quantum vibrations of the relativistic string. What seems almost impossible otherwise, namely quantising gravity, naturally occurs in string theory. String theory therefore serves as a great candidate for a unified theory of all fundamental forces and potentially even a complete theory of physics.

In general one distinguishes between open strings - having two endpoints - and closed strings - having no endpoints. The typical string length can be estimated from the fundamental constants entering string theory [1]. These include the speed of light c (string theory is relativistic), the reduced Planck constant $\hbar$ (string theory is a quantum theory) and the gravitational constant G (string theory includes gravity). They can be combined to form the Planck length of quantum gravity given by

$$l_p = \left(\frac{\hbar G}{c^3}\right)^{\frac{3}{2}} = 1.6 \times 10^{-33}\text{cm} \tag{1}$$

which can be thought of as the typical size of a string. Similarly, the Planck mass is $m_p = (\hbar c/G)^{1/2} = 1.2 \times 10^{19}$ GeV/c^2. At energies significantly lower than this strings can be

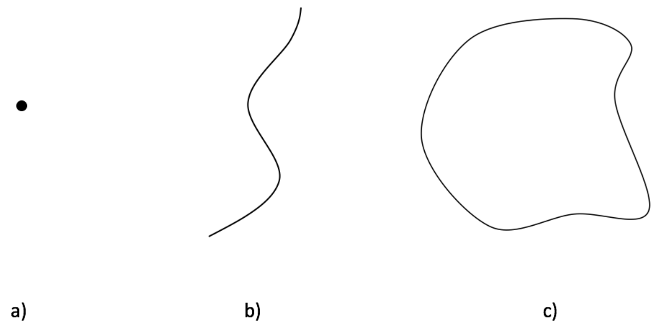

Figure 1: (a) A point particle. (b) An open string. (c) A closed string.

accurately approximated as point particles, therefore explaining the success of quantum field theory for the energies probed at particle accelerators so far.

String theories can be divided into bosonic string theories and superstring theories. Bosonic string theories live in 26 space-time dimensions (as will be shown multiple times in this report) and their vibrations only represent bosons. Obviously a theory containing just bosons can not be an accurate description of nature. Nevertheless bosonic string theories are still worth studying as they are a lot simpler that superstring theories, while at the same time containing most of the important concepts.

All realistic string theories are built on superstrings living in 10 space-time dimensions. Their spectrum includes both bosons and fermions, naturally related to each other by supersymmetry. Supersymmetry is therefore absolutely crucial in finding a string theory that describes our universe.

1.3 A Brief History

Since string theory first arose in the late 60's, there were periods of very high as well as very low interest. It is both interesting and illuminating to see why as this naturally introduces the major discoveries and ideas.

1.3.1 The Early Days

String theory originally arose as an attempt to describe the strong nuclear force in the late 60's. What sparked the interest was the fact that string S-matrix scattering amplitudes exactly matched those found in meson scattering experiments. The inclusion of fermions then quickly led to the discovery of supersymmetric strings. However, physicists slowly started to loose interest as Quantum Chromodynamics was recognised as the correct description of the strong force.

It was not until 1974 that the interest in string theory grew again as it was shown that gravity naturally emerged as one of the string states. For the first time string theory was therefore proposed as a unifying theory of all forces. It had the desirable properties that even though gravity got modified at short distances (at the order of the Planck scale), at large distances it exactly matched Einstein's theory.

1.3.2 The First Superstring Revolution

This was the state of affairs for a couple of years, but again interest in string theory faded as even though various string theories existed, none of them really reproduced the structure of the Standard Model. In 1984 in a landmark paper on "Anomaly Cancellation in Supersymmetric D = 10 Gauge Theory and Superstring Theory" [2] Michael Green and John Schwarz showed how to cancel mathematical inconsistencies. This set of what later became known as the First Superstring Revolution and after the dust had settled string theorists were left with five consistent 10-dimensional superstring theories [3].

10-dimensional theories can be reconciled with our notion of four dimensional spacetime by assuming that the six extra spatial dimensions are compactified. This means that at every point in our 4-dimensional spacetime there exists a compactified 6-dimensional space that is too small to be resolved by experiment. While the simplest compactification for those would be a 6-dimensional torus, the string dynamics restrict those spaces to take more complicated forms, so called "Calabi-Yau" manifolds. Even though these exhibit features very similar to the Standard model, the interest in string theory became weaker yet again. Main reason was the fact that at this point researchers completely lacked an understanding of non-perturbative effects which prevented analysing a realistic vacuum of string theory.

1.3.3 The Second Superstring Revolution

Things took a sharp turn around 1995 with the beginning of the "second super string revolution". The discovery of "dualities", special symmetries relating different theories to each other, finally allowed theorists to go beyond the perturbation expansion of string theory and probe non perturbative features. The three main results of this discovery lead by Edward Witten [4] and others [5], [6], [7] were the following:

- *The five 10-dimensional string theories are related to each other by dualities.*

These dualities are S-duality and T-duality. As T-duality is one of the main subjects of this report, it it briefly explained here. T-duality states that if one theory has a compactified dimension of radius R_A and the other of radius R_B they are equivalent to each other if

$$R_A R_B = (l_p)^2, \tag{2}$$

where l_p is the Planck length [8]. Without going into detail the reason for this duality is that what is interpreted as a momentum excitation in one theory is interpreted as a winding-mode excitation in the T-dual theory and if (2) is obeyed the total string energy is the same in both theories.

In fact going beyond the dualities it was shown that all 10-dimensional theories are just perturbative expansions of an underlying theory $\mathcal{U}$.

- *The theory $\mathcal{U}$ also has an 11-dimensional solution called "M-theory".*

Crucially the five 10-dimensional string theories can be derived from M-theory which has 11-dimensional supergravity as its low energy limit.

- *U theory allows the existence of extended nonperturbative excitations usually referred to as "p-branes" where p stands for the number of spatial dimensions of the object.*

In 1995 Joseph Polchinski [9] identified probably the most important objects of this type as D-branes which are physical objects that constrain the motion of open-string endpoints onto themselves. Studying those quickly suggested new symmetries in M-theory. The most famous is the AdS - CFT correspondence that was introudced by Juan Maldacena in 1997. Broadly speaking as a holographic theory it relates physical laws in a volume to different physical laws on the surface of the volume. In the case of the Ads-CFT correspondence string theory including gravity inside anti-de Sitter spaces (Ads) is related to standard particle physics described by conformal field theories (CFT) on the surface of anti-de Sitter spaces [10].

1.3.4 Present Day Research and Outlook

In the last two decades research in string theory has had applications for a variety of fields ranging from particle physics to understanding black hole entropy and the cosmology of the early universe.

Advances in particle physics have used the Ads-CFT correspondence to transfer problems of certain 4-dimensional gauge theories to their equivalent in string theory which are often easier to handle mathematically. Specifically, this helped understanding the hydrodynamical properties of the quark-gluon plasma created by colliding gold nuclei in heavy ion colliders [11].

String theory also provides a statistical mechanics interpretation for black holes [12]. Usually in statistical mechanics properties such as entropy and temperature are determined by the number of ways one can assemble the constituent parts of the system. However, this interpretation can not be applied to black holes as understood through Einstein's gravitation as those seem to have too few, if any, constituents. String theory in contrast understands certain black holes as a controlled assembly of strings and D-branes, therefore exactly allowing for a statistical mechanics interpretation.

Finally string theory can be used to study the inflation period at the very beginning of our universe. In those regimes classical general relativity breaks down, but string theory can still be applied. String theory can therefore help answer questions on the origin of our universe and thereby solve one of humanities oldest mysteries.

Despite those interesting research applications, doubts have arisen within the physics community and even among string theorists. String theory is criticised for its lack of disprovable predictions, going so far as to suggest that string theory can not be regarded as a true science. However, history has shown numerous times that whenever it was expected the least, a ground breaking discovery would pave the way for another ten years of great string theory research. And who knows maybe the third superstring revolution is just around the corner.

2 Classical Particle and String Dynamics

This section serves as an introduction to the mathematical side of string theory and lays the ground work for all the following sections. It starts with an analysis of the point-particle action and then generalises the ideas to obtain the simplest string action, the Nambu-Goto action. The Nambu-Goto action is subsequently analysed in terms of its equations of motion and symmetries. By introducing an independent worldsheet metric it is then replaced by the more useful Polyakov action. Using the local gauge symmetries, the flat gauge is fixed in which the equations of motion take the form of the two-dimensional wave equations supplemented by an infinite number of additional constraints corresponding to the residual conformal symmetry.

2.1 Particle Dynamics

Before investigating the dynamics of strings it is useful to analyse the familiar motion of a massive relativistic point particle as the techniques applied here readily generalise to the more complicated cases. Throughout the rest of this paper natural units will be used where $c = \hbar = G = 1$.
Consider a particle that is propagating in d-dimensional space-time with coordinates $(t, \vec{x}) = (x^0, x^1, ..., x^{d-1})$. As it propagates it traces out a one-dimensional world-line that can be described by the function $x^\mu(\tau)$. In flat Minkowski space an infinitesimal segment of this worldline is given by

$$ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu \tag{3}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}. \tag{4}$$

The action for the particle of mass m is given by the total length of the trajectory traced out in spacetime [1]:

$$S[x] = -m \int ds = -m \int d\tau \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \tag{5}$$

where a dot represents a derivative with respect to τ. This action is Poincaré-invariant, meaning that it is both Lorentz invariant and translational invariant. In addition to this global symmetry the action is reparameterisation invariant. This means that the action is unchanged as $\tau \to \tau(\tilde{\tau})$ under the requirement that the function $\tau(\tilde{\tau})$ is smooth and $\frac{d\tau}{d\tilde{\tau}} > 0$. It can be seen directly from

$$d\tau\sqrt{-\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}} = d\tau\sqrt{-\frac{dx^\mu}{d\tilde{\tau}}\frac{dx_\mu}{d\tilde{\tau}}\left(\frac{d\tilde{\tau}}{d\tau}\right)^2} = d\tilde{\tau}\sqrt{-\frac{dx^\mu}{d\tilde{\tau}}\frac{dx_\mu}{d\tilde{\tau}}}. \tag{6}$$

This property can be used to set $\tau = x^0 = t$ which is called the static gauge. In this gauge the point particle action (5) becomes

$$S = -m\int\sqrt{1-v^2}dt \tag{7}$$

where $\vec{v} = \frac{d\vec{x}}{dt}$. Demanding that the action is stationary under arbitrary variations $\vec{x}(t)$ one obtains the familiar Euler-Lagrange equations

$$\frac{d\vec{p}}{dt} = 0\ , \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2}} \tag{8}$$

for a free, massive, relativistic particle.
Even though (5) describes the relativistic point particle a more useful action is achieved by introducing a worldline metric $g_{\tau\tau}(\tau)$ and defining $e = \sqrt{-g_{\tau\tau}}$ such that the new action is

$$S' = \frac{1}{2}\int d\tau\left(e^{-1}\dot{x}^2 - em^2\right), \tag{9}$$

where $\dot{x}^2 = \dot{x}^\mu\dot{x}^\nu\eta_{\mu\nu}$ [13]. It can be checked that this action is equivalent to the original action (5) by inserting the equation of motion of e

$$\dot{x}^2 + e^2m^2 = 0 \tag{10}$$

into (9) to recover the original action. S' has the same symmetries as S, Poincaré-invariance and reparameterisation invariance. However, it is more convenient as it also applies to the case of massless particles and is easier to quantise in path integral form because of the absence of the square root.

2.2 String Dynamics

In the previous subsection a point particle, a 0-dimensional object, was analysed in terms of the 1-dimensional worldline it traces out in spacetime. This concept can be easily generalised to a 1-dimensional object, a string, being described by a 2-dimensional worldsheet. An open string traces out a strip in spacetime, while a closed string traces out a tube. One can parameterise a worldsheet in terms of two parameters

$$\left(\xi^0, \xi^1\right) = (\tau, \sigma)\,, \tag{11}$$

where τ is timelike and σ is spacelike. A mapping function $X^\mu(\tau,\sigma)$ which will be called the string coordinates maps the parameters in parameter space onto the worldsheet in spacetime. For closed strings these string coordinates are subject to periodic boundary conditions

$$X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma). \tag{12}$$

2.2.1 Nambu-Goto String Action

Analogoues to the point particle action a key property of the string action is reparameterisation invariance. This has to be satisfied as only the embedding of the worldsheet in spacetime has physical meaning, but not the coordinates that are chosen to parameterise it. In the one dimensional case this was achieved by making the action proportional to the length of the worldline. The obvious generalisation of this is to make the string action proportional to the area of the worldsheet. The area of the worldsheet can be neatly expressed in terms of the parameters ξ_i by introducing the induced metric $h_{\alpha\beta}$ on the worldsheet which is the pull-back of the flat metric in Minkowski space [13]

$$h_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\nu}{\partial \xi^\beta}\eta_{\mu\nu}. \tag{13}$$

The action proportional to the area of the worldsheet is then given by

$$S = -T\int d\xi^2\,\sqrt{-h}, \tag{14}$$

where T is a constant of proportionality that can be identified as the string tension [14] and $h = \det h_{ab}$. This action can be written more explicitly by writing out the induced metric as

$$h_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{pmatrix} \tag{15}$$

where $\dot{X}^\mu = \partial X^\mu/\partial\tau$ and $X'^\mu = \partial X^\mu/\partial\sigma$ and a dot-product and a square imply contraction with the Minkowski metric $\eta_{\mu\nu}$. Inserting the entries of the determinant one obtains the Nambu-Goto action

$$S_{NG} = -T\int d\tau d\sigma\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2\,(X')^2}. \tag{16}$$

An intuitive way to check that the worldsheet area computed is correct is by considering how a small area element in parameter space is transformed under the mapping $X^\mu(\tau,\sigma)$ [11]. A small rectangle of sides $d\tau$ and $d\sigma$ is mapped onto a parallelogram spanned by the vectors dv_1^μ and dv_2^μ which are related to $d\tau$ and $d\sigma$ via

$$dv_1^\mu = \frac{\partial X^\mu}{\partial\tau}d\tau\,, \quad dv_2^\mu = \frac{\partial X^\mu}{\partial\sigma}d\sigma. \tag{17}$$

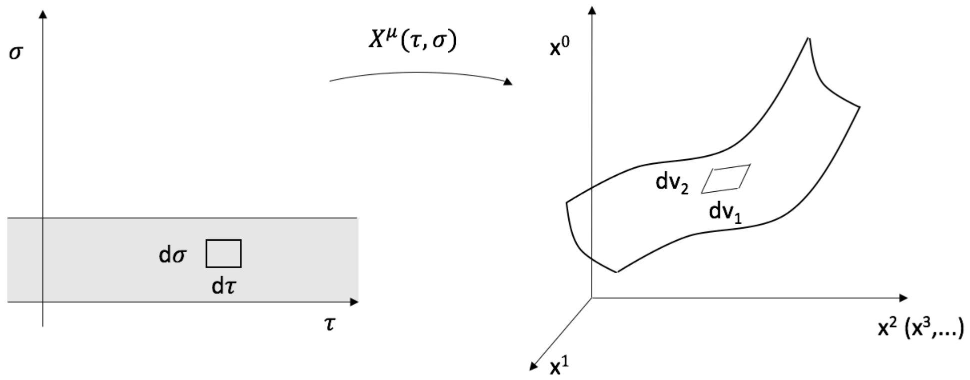

Figure 2: The string coordinates $X^\mu(\tau, \sigma)$ map the parameters τ and σ onto the worldsheet in spacetime.

Elementary geometry shows that the area of the parallelogram is

$$dA = \sqrt{(dv_1 \cdot dv_1)(dv_2 \cdot dv_2) - (dv_1 \cdot dv_2)^2}. \tag{18}$$

Inserting the expressions from (17) into (18) one finds that the integrand takes the same form as in the Nambu-Goto action (16).

2.2.2 Equations of Motion

Proceeding in the same manner as with the relativistic point particle the action of the string can be varied to find the equations of motion. The variation is found to be

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \left[\delta X^\mu P^\sigma_\mu\right]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \delta X^\mu \left(\frac{\partial P^\tau_\mu}{\partial \tau} + \frac{\partial P^\sigma_\mu}{\partial \sigma}\right) \tag{19}$$

where the generalised momenta are

$$\begin{aligned} P^\tau_\mu &= -T\frac{(\dot{X} \cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X}^2)(X')^2}} \\ P^\sigma_\mu &= -T\frac{(\dot{X} \cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X}^2)(X')^2}}. \end{aligned} \tag{20}$$

As the variation of the action has to vanish for all variations of the motion δX^μ both terms on the right-hand side of (19) have to vanish independently. The second will be 0 if

$$\frac{\partial P^\tau_\mu}{\partial \tau} + \frac{\partial P^\sigma_\mu}{\partial \sigma} = 0 \tag{21}$$

which is the equation of motion for the relativistic string. The first term on the right-hand side of (19) will always vanish for closed strings due to periodicity (12). For open strings it can be set to zero by imposing one of two natural boundary conditions at the endpoints σ^*. The first are Dirichlet boundary conditions

$$\left.\frac{\partial X^\mu}{\partial \tau}\right|_{\sigma^*} = 0 \qquad , \mu \neq 0 \tag{22}$$

that fix the string position throughout the motion, thereby causing the variation δX^μ to vanish at the boundaries. The value $\mu = 0$ must be excluded because as τ flows, time must flow. The second are free endpoint conditions

$$P^\sigma_\mu(\tau, \sigma^*) = 0 \tag{23}$$

which always apply for $\mu = 0$.

2.2.3 Symmetries of the Nambu-Goto Action

The Nambu-Goto action has two main symmetries, transformations of $X^\mu(\xi)$ under which the action is invariant, $S_{NG}[X'] = S_{NG}[X]$ [15]. These are:

1. The D-dimensional Poincaré-group

$$X'^\mu(\tau, \sigma) = \Lambda^\mu{}_\nu X^\nu(\tau, \sigma) + a^\mu \tag{24}$$

   where $\Lambda^\mu{}_\nu$ is a Lorentz-transformation and a^μ is a translation. This is a global symmetry as the parameters $\Lambda^\mu{}_\nu$ and a^μ labelling the symmetry transformation do not depend on the worldsheet coordinates (τ, σ).

2. Reparameterisation invariance, sometimes called diffeomorphism invariance. For new coordinates $(\tau'(\tau, \sigma), \sigma'(\tau, \sigma))$ the transformation is

$$X'^\mu(\tau', \sigma') = X^\mu(\tau, \sigma). \tag{25}$$

   This is a local gauge symmetry reflecting the redundancy in the description of the worldsheet as different parameterisations have no physical meaning [13].

2.2.4 Polyakov Action

The Nambu-Goto action (16) is the equivalent of the particle action (5) in the sense that it involves derivatives under a square root making it difficult to quantise in a path integral form [15]. Just like the point particle action it can be simplified by introducing an independent worldsheet metric $\gamma_{\alpha\beta}(\xi)$. Even though this was first done by Brink, Vecchia and Howe [16] the resulting action is commonly know as the Polyakov action,

$$S = -\frac{T}{2}\int d\xi^2 \sqrt{\gamma}\gamma^{\alpha\beta}h_{\alpha\beta} = -\frac{T}{2}\int d\xi^2 \sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{26}$$

as he was the first to realise its significance for quantisation. Here $\gamma = |\mathrm{det}\gamma_{\alpha\beta}|$ and $h_{\alpha\beta}$ is the induced metric (13). The equation of motion by varying the Polyakov action with respect to $\gamma^{\alpha\beta}$ is

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\mu\nu}\partial_\mu X \cdot \partial_\nu X = 0 \tag{27}$$

where $T_{\alpha\beta}$ is the stress-energy tensor defined as

$$T_{\alpha\beta} = -\frac{2\pi}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma^{\alpha\beta}}. \tag{28}$$

One can rewrite (27) as

$$\gamma_{\alpha\beta} = f(\tau,\sigma)\partial_\alpha X \cdot \partial_\beta X = f(\tau,\sigma)h_{\alpha\beta} \tag{29}$$

where $f(\tau,\sigma)$ is defined as

$$f^{-1} = \frac{1}{2}\gamma^{\mu\nu}\partial_\mu X \cdot \partial_\nu X. \tag{30}$$

Inserting the relation between the worldsheet metric $\gamma^{\alpha\beta}$ and the induced metric $h_{\alpha\beta}$ (29) into the Polyakov action (26) the function f drops out of the equation and one recovers the Nambu-Goto action (16), therefore proving that they are equivalent classically.
The Polyakov action has the following symmetries:

1. D-dimensional Poincaré-invariance:
$$\begin{aligned} X'^\mu(\tau,\sigma) &= \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + a^\mu \\ \gamma'_{\alpha\beta}(\tau,\sigma) &= \gamma_{\alpha\beta}(\tau,\sigma) \end{aligned} \tag{31}$$

2. Reparameterisation invariance:
$$\begin{aligned} X'^\mu(\tau',\sigma') &= X^\mu(\tau,\sigma) \\ \gamma'_{\mu\nu}(\tau',\sigma') &= \frac{\partial\sigma^\alpha}{\partial\sigma'^\mu}\frac{\partial\sigma^\beta}{\partial\sigma'^\nu}\gamma_{\alpha\beta}(\tau,\sigma) \end{aligned} \tag{32}$$

3. Two-dimensional Weyl invariance:
$$\begin{aligned} X'^\mu(\tau,\sigma) &= X^\mu(\tau,\sigma) \\ \gamma'_{\alpha\beta}(\tau,\sigma) &= e^{2\rho(\tau,\sigma)}\gamma_{\alpha\beta}(\tau,\sigma) \end{aligned} \tag{33}$$

   for arbitrary $\rho(\tau,\sigma)$.

Compared to the Nambu-Goto action the Polyakov action has an additional symmetry, Weyl invariance. It is a gauge symmetry and states that the action is unchanged under a local rescaling of the metric $\gamma_{\alpha\beta}$. Weyl-invariance is also the reason why the function f cancels when inserting (29) into the Polyakov action (26).

2.2.5 Fixing a Gauge

The two local symmetries can be used to fix a gauge in which the metric γ_{ab} takes a particularly simple form. As it is a symmetric 2×2 matrix it originally depended on three functions. However, the three degrees of freedom (two from 2-dimensional reparameterisation invariance and one from Weyl invariance) can be used to set the metric equal to the flat metric

$$\gamma_{ab} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{34}$$

Substituting this gauge into the Polyakov action classically [8] gives

$$S = -\frac{T}{2} \int d^2\xi \eta^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X. \tag{35}$$

Varying X^μ one finds that the equation of motion is just the free two-dimensional wave equation:

$$\dot{X}^2 - X'^2 = 0. \tag{36}$$

Also substituting the gauge choice into the equation of motion for γ_{ab} (27) gives

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\sigma}\partial_\rho X \cdot \partial_\sigma X = 0. \tag{37}$$

This is equivalent to the two constraints

$$\begin{aligned} T_{01} &= \dot{X} \cdot X' = 0 \\ T_{00} = T_{11} &= \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0. \end{aligned} \tag{38}$$

Thus in the conformally flat gauge the equations of motion of the string is the free wave equation (36) subject to the Virasoro constraints (38).

2.2.6 Mode Expansions

In order to solve the equations of motion (36) it is best to change to light-cone coordinates

$$\xi^\pm = \tau \pm \sigma \tag{39}$$

such that then the equation of motion simply becomes

$$\partial_+ \partial_- X^\mu = 0. \tag{40}$$

The general solution to this equation is

$$X^\mu = X_L^\mu(\xi^+) + X_R^\mu(\xi^-) \tag{41}$$

where X_L^μ and X_R^μ are arbitrary functions describing left and right moving waves respectively. For closed strings obeying $X^\mu(\sigma,\tau) = X^\mu(\sigma+2\pi,\tau)$ the solution can be expanded as a Fourier series [13]

$$\begin{aligned} X_L^\mu(\xi^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \xi^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\xi^+} \\ X_R^\mu(\xi^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \xi^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-in\xi^-}. \end{aligned} \tag{42}$$

where the "Regge slope" α' was introduced which is related to the string tension by $\alpha' = \frac{1}{2\pi T}$. Even though X_L^μ and X_R^μ are not periodic by themselves due to the ξ^+ and ξ^- outside of the infinite sum, their sum X^μ is periodic as required as the σ's outside of the infinite sum cancel. Reality of the string coordinates X^μ demands that the integration constants x^μ and p^μ are real and

$$\alpha_n^\mu = (\alpha_{-n}^\mu)^* \quad , \quad \tilde{\alpha}_n^\mu = (\tilde{\alpha}_{-n}^\mu)^*. \tag{43}$$

Furthermore, integrating X^μ and $\dot{X}^\mu$ over $\sigma \in [0, 2\pi]$ one finds that x^μ and p^μ are indeed the center of mass position and momentum.

2.2.7 Revisiting the Constraints

In addition to the wave equation the constraints (37) have to be written in terms of light cone coordinates. Using the general transformation rule for tensors under reparameterisation, equations (37) become

$$T_{++} = \frac{1}{2}(T_{00} + T_{01}) = \partial_+ X \cdot \partial_+ X = 0 \tag{44}$$

and

$$T_{--} = \frac{1}{2}(T_{00} - T_{01}) = \partial_- X \cdot \partial_- X = 0. \tag{45}$$

One can now evaluate those constraints explicitly to see what conditions they impose on the momenta p^μ and on the Fourier modes α_n^μ and $\tilde{\alpha}_n^\mu$.

$$\begin{aligned} \partial_- X^\mu = \partial_- X_R^\mu &= \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \alpha_n^\mu e^{-in\xi^-} \\ &= \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-in\xi^-} \end{aligned} \tag{46}$$

where the sum in the second line is over all integers n and α_0^μ is defined as

$$\alpha_0^\mu \equiv \sqrt{\frac{\alpha'}{2}} p^\mu. \tag{47}$$

Squaring (46), the constraint arising from T_{--} now reads

$$(\partial_- X)^2 = \frac{\alpha'}{2} \sum_{m,n} \alpha_m \cdot \alpha_{n-m} e^{-in\xi^-} = \alpha' \sum_n L_n e^{-in\xi^-} = 0 \tag{48}$$

where L_n was defined as the sum of the oscillator modes,

$$L_n \equiv \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m. \tag{49}$$

Repeating the same procedure for left-moving modes one can define $\tilde{L}_n$ as

$$\tilde{L}_n \equiv \frac{1}{2} \sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m \tag{50}$$

where the zero mode was defined to be

$$\tilde{\alpha}_0^\mu \equiv \sqrt{\frac{\alpha'}{2}} p^\mu = \alpha_0^\mu. \tag{51}$$

As can be seen from (48) L_n (and similarly $\tilde{L}_n$) have been defined as the Fourier modes of the constraints. Because all modes are independent of each other, the classical solutions (42) therefore obey the infinite number of constraints

$$L_n = \tilde{L}_n = 0 \quad n \in \mathbf{Z}. \tag{52}$$

These constraints corresponds to the residual symmetry, known as the conformal symmetry, that is still present after the worldsheet metric is set equal to the flat metric. This is because there still exist certain combinations of reparameterisation invariance and Weyl-rescaling that preserve the gauge choice of the flat metric [14].

Examining the constraints of L_0 and $\tilde{L}_0$ explicitly one finds that they include the square of the space time momentum p_μ which in Minkowski space is equal to the square of the rest mass of a particle, $p_\mu p^\mu = -M$. One therefore finds a formula for the effective mass of a string in terms of the excited oscillator modes,

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n} \tag{53}$$

The fact that the invariant mass is given as a sum over the right-moving oscillator modes α_n as well as a sum over the left-moving oscillator modes $\tilde{\alpha}_n$ is known as level matching.

3 Old Covariant Quantisation

In general when faced with the task of quantising a gauge theory one has one of two choices to make. Either one can start off by quantising the system and then imposing the constraints as operator equations on the physical states. Or the reverse order can be taken: one first imposes the constraints on the system to find physical solutions and then proceeds to quantise them. In this thesis the first approach will be taken to quantise the string which is known as "Old Covariant Quantisation". Specifically this involves treating all fields X^μ as operators and invoking equal-time commutation relations. The string spectrum that emerges resembles that of two simple harmonic oscillators and physical states have to obey the equivalent of the classical constraints (52). However, a problem with this Hilbert space is the emergence of negative norm states. After calculating the Virasoro algebra, the physicality conditions are applied to the ground state and the first and second excited state. Insisting that those states have positive norm and conserve conformal symmetry restrictions on the number of space-time dimensions and the normal ordering constant are found.

3.1 Commutation Relations

As was shown in section 2.2.5 in the flat gauge the action describing the string dynamics takes the form

$$S = -\frac{T}{2}\int d^2\xi \partial^\alpha X \cdot \partial_\alpha X. \tag{54}$$

It is supported by the two constraints

$$\dot{X} \cdot X' = \dot{X}^2 + X'^2 = 0 \tag{55}$$

and appropriate boundary conditions for the open or closed string. As a first step the string coordinates X^μ and its conjugate momenta Π_μ given by

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\tau X^\mu\right)} = -\frac{T}{2} 2\partial^\tau X_\mu = T\partial_\tau X_\mu = T\dot{X}_\mu \tag{56}$$

are promoted to operator valued fields. As usual in quantum field theory those must obey the canonical equal-time commutation relations

$$[X^\mu(\sigma,\tau), \Pi_\nu\left(\sigma',\tau\right)] = i\delta\left(\sigma-\sigma'\right)\delta^\mu_\nu \tag{57}$$

$$[X^\mu(\sigma,\tau), X^\nu\left(\sigma',\tau\right)] = [\Pi_\mu(\sigma,\tau), \Pi_\nu\left(\sigma',\tau\right)] = 0. \tag{58}$$

One can use these in combination with the mode expansion (42) to find the commutation relations for the Fourier modes x^μ, p^μ, α_m^μ and $\tilde{\alpha}_m^\mu$ (see Appendix A). They are given by

$$[x^\mu, p_\nu] = i\delta_\nu^\mu \tag{59}$$
$$[\alpha_n^\mu, \alpha_m^\nu] = [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0} \tag{60}$$

with all others equal to zero.
In addition to the expected commutation relation for the position and the momentum of the center of mass of the string, one finds commutation relations similar to those of a harmonic oscillator. Indeed, by redefining

$$a_n^\mu = \frac{\alpha_n^\mu}{\sqrt{n}} \quad , \quad a_n^{\mu\dagger} = \frac{\alpha_{-n}^\mu}{\sqrt{n}} \quad \text{for} \quad n > 0, \tag{61}$$

(and similarly for $\tilde{a}_n^\mu$ and $\tilde{a}_n^{\mu\dagger}$) one can make this correspondence explicit. It emphasises that one obtains two infinite towers of creation and annihilation operators corresponding to right-moving modes and left-moving modes.
Using the commutation relations (60) one can build a Fock space for the string states. This is done by introducing a ground state $|0\rangle$ that is defined to obey

$$\alpha_n^\mu|0\rangle = \tilde{\alpha}_n^\mu|0\rangle = 0 \qquad \text{for} \quad n > 0. \tag{62}$$

Besides the oscillator level, string states have a second degree of freedom, the center of mass momentum. This must be specified as well to fully determine a string state, e.g. the ground state with momentum p^μ is denoted as $|0; p_\mu\rangle$. Any state can now be obtained by acting with any number of creation operators α_n^μ and $\tilde{\alpha}_n^\mu$ with $n > 0$ on the ground state,

$$\left(\alpha_{-1}^{\mu_1}\right)^{n_{\mu_1}} \left(\alpha_{-2}^{\mu_2}\right)^{n_{\mu_2}} \ldots \left(\tilde{\alpha}_{-1}^{\nu_1}\right)^{n_{\nu_1}} \left(\tilde{\alpha}_{-2}^{\nu_2}\right)^{n_{\nu_2}} \ldots |0; p\rangle. \tag{63}$$

3.2 Ghosts

When exploring the Fock space constructed one quickly encounters the problem that some states have negative norm. These states are usually referred to as 'ghosts'. They arise because of the minus sign coming from the Minkowski metric in the time components of the commutation relations, $\left[\alpha_n^0, \alpha_n^{0\dagger}\right] = -n$. For $n > 0$ a state $\alpha_n^{0\dagger}|0\rangle$ therefore has negative norm as

$$\left\langle 0 \left| \alpha_n^0 \alpha_n^{0\dagger} \right| 0 \right\rangle = \left\langle 0 \left| \left[\alpha_n^0, \alpha_n^{0\dagger}\right] \right| 0 \right\rangle = -n \tag{64}$$

As those states would have negative probabilities associated with them, one has to ensure that they can not be produced in any physical process. The space of physically allowed states is therefore a subspace of the complete Fock space, excluding the negative-norm states.

3.3 Constraints

As was shown in section 2.2.7 the classical constraints lead to the vanishing of the Fourier modes of the stress energy tensor, $L_n = \tilde{L}_n = 0$, where

$$L_n = \frac{1}{2}\sum_m \alpha_{n-m} \cdot \alpha_m. \tag{65}$$

When passing to the quantum theory the α_m are promoted to operators, therefore potentially causing ordering ambiguities. From the commutation relations one can see that for $n \neq 0$, the operators α_m and α_{n-m} commute, so that in this case L_n is clearly defined. In the case, however, that $n = 0$ those ordering ambiguities do arise. Those ambiguities manifest themselves in terms of a constant that arises when commuting the α_n^μ past each other in L_0. Therefore, at this stage one simply defines L_0 to be given by the normal-ordered expression

$$L_0 \equiv \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \tag{66}$$

and includes a constant a in every expression of L_0.
Instead of imposing all classical constraints as operator equations on the Hilbert space it is sufficient to require that the matrix elements of L_n and $\tilde{L}_n$ vanish when inserted between two physical states $|\text{phys}\rangle$ and $|\text{phys}'\rangle$,

$$\langle \text{phys}'|L_n|\text{phys}\rangle = \langle \text{phys}'|\tilde{L}_n|\text{phys}\rangle = 0. \tag{67}$$

Because $L_n^\dagger = L_{-n}$, (67) is always fulfilled if

$$L_n|\text{phys}\rangle = \tilde{L}_n|\text{phys}\rangle = 0 \quad for \quad n > 0 \tag{68}$$

and including a for the expression of L_0 and $\tilde{L}_0$

$$(L_0 - a)|\text{phys}\rangle = (\tilde{L}_0 - a)|\text{phys}\rangle = 0. \tag{69}$$

In a later section these conditions will be systematically applied to the vacuum as well as a general first and second order excited state, giving restrictions on the dimensions of space time and the constant a.
Before calculating the value of a it is useful to understand its physical significance. By combining (69) with the equation for the classical level matching (53) one obtains

$$M^2 = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m\right) = \frac{4}{\alpha'}\left(-a + \sum_{m=1}^{\infty} \tilde{\alpha}_{-m} \cdot \tilde{\alpha}_m\right). \tag{70}$$

This shows that a shifts the mass spectrum of the string. Additionally, the level matching in the quantum theory implies that there is an equal number of left and right moving modes on a string.

3.4 Virasoro Algebra

One can evaluate the commutator $[\alpha^\nu_m, L_n]$ by explicitly inserting the expression for L_n (see Appendix B). One finds that

$$[\alpha^\nu_m \, , \, L_n] = m \, \alpha^\nu_{m+n} \tag{71}$$

holds for all m and n. One can now use this relation to evaluate the commutator $[L_m, L_n]$. For this note that

$$L_m = \frac{1}{2} \sum : \alpha^\mu_{m-k} \alpha^\mu_k : \qquad \forall \, m \in Z. \tag{72}$$

where :: imply normal ordering, such that α_m are moved to the right for $m > 0$ and to the left for $m < 0$. This holds true because L_0 was originally defined to be normal-ordered and the α_m in the other L_n commute and can therefore be brought into normal-ordered form. Dropping space-time indices for convenience, the commutator of L_m and L_n can be calculated.[1] Taking particular care of the normal ordering one finds [17],

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\Big(\sum_{k\geq 0} \alpha_{m-k}\alpha_k L_n + \sum_{k<0} \alpha_k \alpha_{m-k} L_n - \sum_{k\geq 0} L_n \alpha_{m-k}\alpha_k - \sum_{k<0} L_n \alpha_k \alpha_{m-k} \Big) \\
&= \frac{1}{2}\Big(\sum_{k\geq 0} \alpha_{m-k}\left(L_n \alpha_k + k\alpha_{k+n}\right) - \left(\alpha_{m-k} L_n - (m-k)\,\alpha_{m+n-k}\right)\alpha_k \\
&\qquad + \sum_{k<0} \alpha_k \left(L_n \alpha_{m-k} + (m-k)\,\alpha_{m-k+n}\right) - \left(\alpha_k L_n - k\alpha_{n+k}\right)\alpha_{m-k} \Big) \\
&= \frac{1}{2}\left(\sum_{k\geq 0} \Big((m-k)\alpha_{m+n-k}\alpha_k + k\alpha_{m-k}\alpha_{n+k} \Big) + \sum_{k<0} \Big(k\alpha_{k+n}\alpha_{m-k} + (m-k)\alpha_k\alpha_{m+n-k} \Big) \right)
\end{aligned} \tag{73}$$

If $m + n \neq 0$ then the α_m commute in all four terms. One can therefore switch the order of the α_m in the last two terms and then add pairwise terms one and four and two and three to obtain two sums over all k.

$$(73) = \frac{1}{2}\left(\sum_{k=-\infty}^{\infty} (m-k)\alpha_{m+n-k}\alpha_k + \sum_{k=-\infty}^{\infty} k\alpha_{m-k}\alpha_{n+k} \right) \tag{74}$$

Replacing k with $k - n$ one finds the commutator as

$$[L_m, L_n] = \frac{1}{2} \sum_{k=-\infty}^{\infty} (m-n)\alpha_{m+n-k}\alpha_k = (m-n)L_{m+n}. \tag{75}$$

[1] This calculation was deliberately not relegated to the appendix, as it is instructive to see how the quantum anomaly, that is so crucial to string theory, arises.

In the case that $m+n=0$ the commutators do not vanish and an extra contribution arises as one ensures normal ordering. Assuming $m>0$ (an analogues argument can be made for the case $m<0$) the only term that is not normal ordered is the second term. Here, the order has to be switched for all values $0 \leq k \leq m$. From the commutation relations (60) this gives rise to a factor $m-k$, such that the extra terms is:

$$+\frac{1}{2}\sum_{k=1}^{m} k(m-k)\delta_{m+n}. \tag{76}$$

Using standard formulae for the sum of integers and the sum of squares this sum evaluates to

$$\frac{1}{2}\sum_{k=1}^{m} k(m-k) = \frac{1}{4}m^2(m+1) - \frac{1}{12}m(m+1)(2m+1) = \frac{1}{12}m(m+1)(m-1). \tag{77}$$

Therefore arriving at the very important Virasoro algebra,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}m\left(m^2-1\right)\delta_{m+n}. \tag{78}$$

The only case that was not checked explicitly above is the case $m=n=0$. However, from (73) one can see that in this case the commutator vanishes which is in accordance with the Virasoro algebra.

3.5 State Analysis

After evaluating the commutation relations, one can use those to see what conditions the constraints (68) and (69) impose on a and the space time dimensions. This will be done by applying the L_n on the ground state and the first and second excited state.

3.5.1 Ground State

Starting with the ground state $|0;p\rangle$ one sees that applying L_n with $n>0$ is identically 0 as every term in the sum will include an annihilation operator that annihilates the ground state. Applying L_0, however, one finds

$$(L_0 - a)|0;p\rangle = (\frac{1}{2}\alpha_0^2 + \sum_{n=-\infty}^{\infty} \alpha_{-n}\cdot\alpha_n - a)|0;p\rangle = (\frac{\alpha'}{4}p^2 - a)|0;p\rangle = 0. \tag{79}$$

Therefore the ground state of a string has mass $M^2 = -\frac{4}{\alpha'}a$. Later it will be shown that $a=1$ such that the ground state corresponds to a Tachyon which is a particle that has negative mass-squared. Without going into details the minus sign means that one is expanding around a maximum of the potential for the Tachyon field rather than a minimum. This is problematic as it indicates that there might not exist a time-independent stable solution to bosonic string theory [13]. Despite this obvious flaw bosonic string theory is still worth studying as all the concepts can be carried over directly to super string theory which does not suffer from this problem.

3.5.2 First Excited State

Applying the physicality conditions on a general first excited state $|1\rangle = \xi_\nu \alpha_{-1}^\nu |0;p\rangle$ one finds

$$(L_0 - a)|1\rangle = 0 \qquad \rightarrow \qquad \frac{\alpha'}{4}p^2 = a - 1 \tag{80}$$

$$L_1|1\rangle = 0 \qquad \rightarrow \qquad \xi \cdot p = 0. \tag{81}$$

as shown in Appendix C. Operating with L_n when $n > 1$ on the first excited state will not provide any further constraints as those equations are trivially zero. However, one can obtain an extra condition by considering the norm of the first excited state $|1\rangle$.

$$\begin{aligned}\langle 1|1\rangle &= \langle 0;p|\alpha_1^\mu \xi_\mu^* \xi_\nu \alpha_{-1}^\nu |0;p\rangle \\ &= \xi_\mu^* \xi_\nu \langle 0;p|\alpha_{-1}^\nu \alpha_1^\mu + \eta^{\mu\nu}|0;p\rangle \\ &= \xi^* \cdot \xi\end{aligned} \tag{82}$$

As the first excited state is a physical state its norm must be greater or equal to zero, such that

$$\xi^* \cdot \xi \geq 0. \tag{83}$$

This implies that ξ is space-like or light-like. Because of (81) the momentum p must therefore be time-like or light-like, so that $p^2 \leq 0$. Using (80) this restricts a:

$$a \leq 1. \tag{84}$$

One can go even further than this by looking at the symmetry that is left after choosing a particular value of a. After fixing the gauge to the flat metric there is still a residual symmetry left over. This conformal symmetry is generated by L_{-n} (for $n > 0$). Therefore, when quantising the theory one should ensure that this symmetry is still present. Specifically for the first excited state one should still be able to make an identification between two first excited states $|1\rangle$ and $|1'\rangle$ where $|1\rangle$ is defined as before and $|1'\rangle$ is

$$|1'\rangle = |1\rangle + \gamma L_{-1}|0;p\rangle. \tag{85}$$

This state can be rewritten as

$$\begin{aligned}|1'\rangle &= \xi_\mu \alpha_{-1}^\mu + \frac{\gamma}{2}\sqrt{\frac{\alpha'}{2}} p_\mu \alpha_{-1}^\mu |0;p\rangle \\ &\equiv \xi'_\mu \alpha_{-1}^\mu |0;p\rangle\end{aligned} \tag{86}$$

where ξ'_μ was defined as $\xi'_\mu = \xi_\mu + \frac{\gamma}{2}\sqrt{\frac{\alpha'}{2}}p_\mu$. As this state is physical the same conditions (80) and (81) must still hold. Checking $\xi' \cdot p$ gives

$$\xi'_\mu p^\mu = \xi_\mu p^\mu + \gamma p^2 = \gamma p^2 = \frac{4}{\alpha'}\gamma(a-1) \tag{87}$$

This is only equal to 0 as required if either $\gamma = 0$ or $a = 1$. But $\gamma = 0$ implies that there are no possible identifications, therefore destroying the conformal symmetry. This means

$$\boxed{a = 1}. \tag{88}$$

3.5.3 Second Excited State

Analogous to the first excited state one applies the conditions for a physical state on a general second order excited state

$$|2\rangle = \xi_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|0;p\rangle + \tilde{\xi}_\mu\alpha^\mu_{-2}|0;p\rangle. \tag{89}$$

The full calculations can be found in Appendix C.

$$(L_0 - a)|2\rangle = 0 \qquad \rightarrow \qquad p^2 = -2 \tag{90}$$

$$L_1|2\rangle = 0 \qquad \rightarrow \qquad \xi_{\mu\nu}p^\nu + \tilde{\xi}_\mu = 0 \tag{91}$$

$$L_2|2\rangle = 0 \qquad \rightarrow \qquad \xi_{\mu\nu}\eta^{\mu\nu} + 2\tilde{\xi}_\mu p^\mu = 0 \tag{92}$$

Again using the conformal symmetry argument demand that the sum of a general second order excited state and a second order conformally invariant state is still a physical state. Such a second order conformally invariant state is given by

$$|\chi\rangle = L_{-1}|1;p\rangle + \gamma L_{-2}|0;p\rangle = \pi_\mu L_{-1}\alpha^\mu_{-1}|0;p\rangle + \gamma L_{-2}|0;p\rangle \tag{93}$$

One can expand $|\chi\rangle$ to write it in a form similar to $|2\rangle$

$$\begin{aligned}
|\chi\rangle &= \pi_\mu(\alpha^\mu_{-1}L_{-1} + \alpha^\mu_{-2})|0;p\rangle + \gamma\frac{1}{2}(\alpha^\mu_{-2}\alpha_{0\mu} + \alpha^\mu_{-1}\alpha_{-1\mu} + \alpha^\mu_0\alpha_{-2\mu})|0;p\rangle \\
&= \Big[\pi_\mu(\alpha^\mu_{-1}\frac{1}{2}(\alpha^\nu_{-1}\alpha_{0\nu} + \alpha^\nu_0\alpha_{-1\nu}) + \alpha^\mu_{-2})|0;p\rangle + \gamma\frac{1}{2}(p_\mu\alpha^\mu_{-2} + \alpha^\mu_{-1}\alpha_{-1\mu})\Big]|0;p\rangle \\
&= \Big[(\frac{1}{2}\pi_\mu p_\nu + \frac{\gamma}{2}\eta_{\mu\nu})\alpha^\mu_{-1}\alpha^\nu_{-1} + (\pi_\mu + \frac{1}{2}\gamma p_\mu)\alpha^\mu_{-2}\Big]|0;p\rangle \\
&= \Big[\Delta\xi_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + \Delta\xi_\mu\alpha^\mu_{-2}\Big]|0;p\rangle
\end{aligned} \tag{94}$$

where in the last step $\Delta\xi_{\mu\nu}$ and $\Delta\xi_\mu$ were defined as

$$\Delta\xi_{\mu\nu} = \frac{1}{2}(\pi_\mu p_\nu + \gamma\eta_{\mu\nu}) \qquad \text{and} \qquad \Delta\xi_\mu = \pi_\mu + \frac{1}{2}\gamma p_\mu \tag{95}$$

Now one has to check that

$$\begin{aligned}|2'\rangle = |2\rangle + |\chi\rangle &= \Big[(\xi_{\mu\nu} + \Delta\xi_{\mu\nu})\alpha^{\mu}_{-1}\alpha^{\nu}_{-1} + (\xi_{\mu} + \Delta\xi_{\mu})\alpha^{\mu}_{-2}\Big]|0;p\rangle \\ &\equiv \Big[\xi'_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1} + \xi'_{\mu}\alpha^{\mu}_{-2}\Big]|0;p\rangle\end{aligned} \tag{96}$$

is still a physical state. Rather than checking the physicality conditions (90), (91) and (92) for the primed fields $\xi'_{\mu\nu}$ and ξ'_{μ}, one can use the fact that the physicality conditions are linear and instead check them for $\Delta\xi'_{\mu\nu}$ and $\Delta\xi'_{\mu}$ as this shortens the algebra.
Inserting (95) into (91) one finds

$$\Delta\xi_{\mu\nu}p^{\nu} + 2\Delta\tilde{\xi}_{\mu} = \frac{1}{2}(\pi_{\mu}p_{\nu} + \gamma\eta_{\mu\nu})p^{\nu} + 2\pi_{\mu} + \gamma p_{\mu} = -2\pi_{\mu} + \frac{3}{2}\gamma p_{\mu} = 0 \tag{97}$$

where (90) was used. Multiplying by p^{μ} one finds that

$$\pi_{\mu}p^{\mu} = -6\gamma. \tag{98}$$

Repeating the same for (92) and using (90) and (98) one gets

$$\begin{aligned}\Delta\xi_{\mu\nu}\eta^{\mu\nu} + \Delta\tilde{\xi}_{\mu}p^{\mu} &= \frac{1}{2}(\pi_{\mu}p_{\nu} + \gamma\eta_{\mu\nu})\eta^{\mu\nu} + (\pi_{\mu} + \frac{1}{2}\gamma p_{\mu})p^{\mu} \\ &= \frac{1}{2}\pi_{\mu}p^{\mu} + \frac{\gamma}{2}D + \pi_{\mu}p^{\mu} - 4\gamma \\ &= \left(\frac{D}{2} - 13\right)\gamma = 0\end{aligned} \tag{99}$$

where D is the number of space time dimensions. Again γ should not vanish as this would destroy the conformal symmetry. Therefore one obtains the famous restriction on the number of space time dimensions in bosonic string theory:

$$\boxed{D = 26} \tag{100}$$

3.6 No-Ghost Theorem

Even though this is not shown here it turns out that setting $D = 26$ and $a = 1$ preserves the conformal symmetry for states of all excited levels. This effectively removes all ghost states from the theory as every negative norm state can be identified with a positive or zero norm state by conformal symmetry. The interested reader can find a full proof of this in a paper by Goddard and Thorn [18].

4 BRST Quantisation

A common challenge when working with gauge theories is the emergence of unphysical degrees of freedom in the Lagrangian. They can be removed by fixing a gauge, but the problem is that this destroys gauge invariance. The idea of BRST quantisation is to replace the original symmetry by a new symmetry, the BRST symmetry, which is still present after the gauge is fixed. This is done by introducing new ghost[2] and anti-ghost fields that allow constructing a BRST invariant action. The operator that generates this symmetry is called BRST differential, Q, and has the very important property that it is nilpotent, $Q^2 = 0$. The reason why this is so important is that even though introducing the new fields has greatly enlarged the Hilbert space the subspace corresponding to physical states simply appears as the cohomology of Q.

4.1 BRST Action

While the following is far from a complete derivation of the BRST action, it aims to motivate the different terms and the introduction of the new fields. As stated above the original gauge symmetry is replaced by the BRST symmetry which encompasses the original gauge symmetry. Therefore, looking at the infinitesimal transformations, rather than having reparameterisation invariance given by

$$\delta X^\mu(\xi) = \epsilon^\alpha \partial_\alpha X^\mu \tag{101}$$

where ϵ^α is some small function, let the transformation be

$$\delta_B X^\mu(\xi) = \epsilon c^\alpha \partial_\alpha X^\mu \tag{102}$$

where c^α is a ghost field and δ_B denotes an infinitesimal BRST transformation. In terms of the metric this gives the infinitesimal transformation,

$$\delta\gamma_{mn} = \nabla_m c_n + \nabla_n c_m. \tag{103}$$

Similarly introduce a second ghost field $\tilde{c}$ for the Weyl transformation, such that

$$\delta\gamma_{mn} = \epsilon\tilde{c}\gamma_{mn}. \tag{104}$$

Under a full BRST transformation γ_{mn} therefore changes as

$$\delta_B\gamma_{mn} = \epsilon(\nabla_m c_n + \nabla_n c_m + \tilde{c}\gamma_{mn}). \tag{105}$$

[2] These are not to be confused with the negative norm states appearing in section 3.2. They are simply new fermionic fields necessary to maintain gauge invariance.

Now one can add a gauge fixing term to the Polyakov action. Let the gauge fixing be

$$F_{mn} \equiv \gamma_{mn} - \eta_{mn} = 0. \tag{106}$$

such that

$$S = S_P - \frac{i}{4\pi} \int d^2\xi \sqrt{\gamma} F_{mn} B^{mn} \tag{107}$$

where B^{mn} takes the role of a Lagrange multiplier and therefore $\delta_B B^{mn} = 0$. The presence of $\sqrt{\gamma}$ ensures that the volume element $\int d^2\xi\sqrt{\gamma}$ is invariant under BRST symmetry as it was invariant under normal gauge symmetry which is a special case of BRST symmetry. By the same argument the Polyakov action is also invariant under a BRST transformation. However, the term $F_{mn}B^{mn}$ is not invariant. Therefore in order to make the whole action invariant, one has to add a third term that cancels the contribution from the second term. For this an anti-ghost field b^{mn} is introduced whose variation is defined to be $\delta_B b^{mn} = \epsilon B^{mn}$. Also defining the variation of the gauge fixing $\delta_B F_{mn} = \epsilon \Lambda_{mn}$ the third term should be

$$\frac{i}{4\pi} \int d^2\xi \sqrt{\gamma} \Lambda_{mn} b^{mn}. \tag{108}$$

When the other infinitesimal transformations are given by

$$\delta_B c_m = \epsilon \mathcal{L}_c c_m = \epsilon(c^g \partial_g c_m + c_g \partial_n c^g) \tag{109}$$

$$\delta_B \tilde{c} = \epsilon \mathcal{L}_c \tilde{c} = \epsilon c^g \partial_g \tilde{c} \tag{110}$$

where $\mathcal{L}_c$ is the Lie derivative with respect to c, the whole action

$$S = S_P - \frac{i}{4\pi} \int d^2\xi \sqrt{\gamma} F_{mn} B^{mn} + \frac{i}{4\pi} \int d^2\xi \sqrt{\gamma} \Lambda_{mn} b^{mn} \tag{111}$$

is therefore invariant under a BRST transformation. This is because

$$\delta_B \Lambda_{mn} = \frac{1}{\epsilon} \delta_B \delta_B F_{mn} = 0 \tag{112}$$

by nilpotency of Q.

Now inserting for F_{mn} and Λ_{mn} one gets

$$S = S_P - \frac{i}{4\pi} \int d^2\xi \sqrt{\gamma} (\gamma_{mn} - \eta_{mn}) B^{mn} + \frac{i}{4\pi} \int d^2\xi \sqrt{\gamma} (\nabla_m c_n + \nabla_n c_m + \tilde{c}\gamma_{mn}) b^{mn} \tag{113}$$

Note that both B_{mn} and $\tilde{c}$ have algebraic equations of motion. By construction the B_{mn} equation of motion gives the gauge fixing $\gamma_{mn} - \eta_{mn} = 0$. The equation of motion from $\tilde{c}$ gives $b^{mn}\gamma_{mn} = 0$ such that b_{mn} must be traceless. Inserting those relations the action simplifies to

$$S = S_P + \frac{i}{4\pi} \int d^2\xi \sqrt{\gamma} (\partial_m c_n + \partial_n c_m) b^{mn} = S_P + \frac{i}{2\pi} \int d^2\xi \sqrt{\gamma} \gamma^{\alpha m} \partial_\alpha c^n b_{mn} \tag{114}$$

as the covariant derivatives can be replaced by partial derivatives when working in the flat frame. In light cone coordinates after an integration by parts the ghost action reads

$$S_g = -\frac{i}{\pi}\int d^2\xi (c^+\partial^+ b_{++} + c^-\partial^+ b_{+-} + c^+\partial^- b_{-+} + c^-\partial^- b_{--}). \tag{115}$$

Using the fact that $b_{\alpha\beta}$ is traceless and symmetric further simplifications can be made. Vanishing of the trace implies $-b_{-+} - b_{+-} = 0$. Because of the symmetry however, $b_{+-} = b_{-+}$, which combined gives $b_{+-} = b_{-+} = 0$. Therefore, the ghost action simplifies to

$$S_g = -\frac{i}{\pi}\int d^2\xi (c^+\partial^+ b_{++} + c^-\partial^- b_{--}) = \frac{i}{\pi}\int d^2\xi (c^+\partial_- b_{++} + c^-\partial_+ b_{--}) \tag{116}$$

which agrees with the textbook by Green, Schwarz and Witten [14].

4.2 Stress-Energy Tensor

Using the formula for the world-sheet energy momentum tensor

$$T_{\alpha\beta} = -\frac{2\pi}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma^{\alpha\beta}} \tag{117}$$

one can find the contribution from the ghost action to the total energy-momentum tensor. Starting from the ghost action in the form

$$S_g = \frac{i}{2\pi}\int d^2\xi \sqrt{\gamma}\gamma^{\alpha m}\nabla_\alpha c^n b_{mn} \tag{118}$$

and taking care when varying the Christoffel symbols arising in the covariant derivative as shown in Appendix D, this evaluates to

$$T^{(c)}_{\alpha\beta} = -\frac{i}{2}\Big[(\nabla_\alpha c^\mu)b_{\beta\mu} + (\nabla_\beta c^\mu)b_{\alpha\mu} - \gamma_{\alpha\beta}\gamma^{mn}(\nabla_m c^\mu)b_{n\mu} + c^\nu\nabla_\nu(b_{\alpha\beta})\Big]. \tag{119}$$

For example $T^{(c)}_{--}$ in the conformal gauge is given by

$$\begin{aligned} T^{(c)}_{--} &= -\frac{i}{2}\Big[(\partial_- c^\mu)b_{-\mu} + (\partial_- c^\mu)b_{-\mu} - \gamma_{--}\gamma^{mn}(\partial_m c^\mu)b_{n\mu} + c^\nu\partial_\nu(b_{--})\Big] \\ &= -\frac{i}{2}\Big[2(\partial_- c^+)b_{-+} + 2(\partial_- c^-)b_{--} + c^+\partial_+ b_{--} + c^-\partial_- b_{--}\Big]. \end{aligned} \tag{120}$$

4.3 Equations of Motion, Commutation Relations and Mode Expansions

In order to find the conjugate momenta of the fields one can rewrite the ghost action

$$S_g = \frac{i}{\pi}\int d^2\xi \big[c^+\frac{1}{2}(\partial_\tau - \partial_\sigma)b_{++} + c^-\frac{1}{2}(\partial_\tau + \partial_\sigma)b_{--})\big] \tag{121}$$

and using the definition of the conjugate momenta

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \left(\partial_\tau X^\mu\right)} \tag{122}$$

one can read off the conjugate momenta:

$$\Pi^{(c_+)} = \Pi^{(c_-)} = 0 \tag{123}$$

$$\Pi^{(b_{++})} = \frac{i}{2\pi} c^+ \quad \text{and} \quad \Pi^{(b_{--})} = \frac{i}{2\pi} c^- \tag{124}$$

Using those one finds the equal time anti-commutation relations for the ghost fields

$$\begin{aligned} \Big\{ b_{++}(\sigma,\tau)\,,\, c^+(\sigma',\tau) \Big\} &= 2\pi\delta(\sigma-\sigma') \\ \Big\{ b_{--}(\sigma,\tau)\,,\, c^-(\sigma',\tau) \Big\} &= 2\pi\delta(\sigma-\sigma') \end{aligned} \tag{125}$$

with all others equal to zero.
Varying the ghost action (116) with respect to the four fields, and using integration by parts when varying with respect to b_{++} and b_{--}one immediately finds the four equations of motion

$$\begin{aligned} \partial_- c^+ &= \partial_- b_{++} = 0 \\ \partial_+ c^- &= \partial_+ b_{--} = 0. \end{aligned} \tag{126}$$

Just as in the case of left and right moving coordinates X^μ, c^+ and c^- have independent mode expansions.

$$c^+ = \sum_{n=-\infty}^{+\infty} \tilde{c}_n e^{-in(\tau+\sigma)} \tag{127}$$

$$c^- = \sum_{n=-\infty}^{+\infty} c_n e^{-in(\tau-\sigma)} \tag{128}$$

Similarly,

$$b_{++} = \sum_{n=-\infty}^{+\infty} \tilde{b}_n e^{-in(\tau+\sigma)} \tag{129}$$

$$b_{--} = \sum_{n=-\infty}^{+\infty} b_n e^{-in(\tau-\sigma)} \tag{130}$$

From these one can find the anticommutation relations for b_m and c_m. Note that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} d\sigma e^{im\sigma} b^{++}(\sigma, \tau=0) &= \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} b_n \int_0^{2\pi} d\sigma e^{im\sigma} e^{-in\sigma} \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} b_n 2\pi\delta_{n-m} \\ &= b_m\end{aligned} \tag{131}$$

and similarly for c_m. Therefore,

$$\begin{aligned}\Big\{c_m\,,\,b_n\Big\} &= \int_0^{2\pi}\int_0^{2\pi} d\sigma d\sigma' \frac{1}{4\pi^2} e^{im\sigma} e^{in\sigma'} \big\{b_{++}(\sigma,\tau)\,,\,c^+(\sigma',\tau)\big\} \\ &= \frac{1}{2\pi}\int_0^{2\pi} d\sigma e^{i(m+n)\sigma} = \delta_{m+n}.\end{aligned} \tag{132}$$

Also

$$\Big\{c_m\,,\,c_n\Big\} = \Big\{b_m\,,\,b_n\Big\} = 0. \tag{133}$$

4.4 Extracting Fourier Modes

Ultimately again the goal is to arrive at the Virasoro algebra. For that one needs to extract the ghost contribution to the L_n out of the ghost stress-energy tensor. First however note that one can simplify the ghost stress-energy tensor (120) using the equations of motion (126), such that

$$T^{(c)}_{--} = -\frac{i}{2}\Big[2(\partial_- c^-)b_{--} + c^-\partial_- b_{--}\Big]. \tag{134}$$

Now for the closed string (at $\tau = 0$)

$$L^{(c)}_n = T\int_0^{2\pi} d\sigma e^{-in\sigma} T^{(c)}_{--} = \sum_{m=-\infty}^{\infty} (n-m) : b_{n+m} c_{-m} : \tag{135}$$

as shown in Appendix E.

4.5 Commutation Relations for Fourier Modes

One again has to evaluate the commutator of the ghost contribution to the L_n operators. As before this will be of the type

$$\big[L^{(c)}_m, L^{(c)}_n\big] = (m-n)L^{(c)}_{m+n} + A^c(m)\delta_{m+n}. \tag{136}$$

Just as in the Old Covariant Quantisation one could calculate the anomaly by explicitly evaluating the commutator, carefully ensuring normal ordering during the procedure. This calculation however is very tedious which is why a different approach will be used here that follows [14] and evaluates $A^c(m)$ indirectly.
Looking at (136) for $n = m = 0$ one sees that $A(0) = 0$. Also taking $n = -m$ and comparing to the commutator in reverse order one sees $A(m) = -A(-m)$, so that it is sufficient to evaluate $A(m)$ for positive m. From the Jacobi identity

$$\left[L_k^{(c)}, \left[L_n^{(c)}, L_m^{(c)}\right]\right] + \left[L_n^{(c)}, \left[L_m^{(c)}, L_k^{(c)}\right]\right] + \left[L_m^{(c)}, \left[L_k^{(c)}, L_n^{(c)}\right]\right] = 0 \tag{137}$$

one finds that for $k + n + m = 0$

$$(n-m)A^c(k) + (m-k)A^c(n) + (k-n)A^c(m) = 0. \tag{138}$$

Setting $k = 1$ and $m = -n - 1$ in (138) one can obtain the recursion relation for $A(n)$

$$A^c(n+1) = \frac{(n+2)A^c(n) - (2n+1)A^c(1)}{(n-1)}. \tag{139}$$

All $A^c(n)$ can be determined in terms of $A^c(1)$ and $A^c(2)$ and in fact the general solution can be written as

$$A^c(m) = c_3 m^3 + c_1 m \tag{140}$$

where c_1 and c_3 are constants.
These two constants can be evaluated by calculating the expectation value of (136) for $n = 1, 2$ for a suitably chosen state. The most convenient choice is the ground state $|0;0\rangle$. Evaluating the matrix element $\langle 0;0|[L_1^{(c)}, L_{-1}^{(c)}]|0;0\rangle$ one gets $A^c(1) = -2$. Similarly for $\langle 0;0|[L_2^{(c)}, L_{-2}^{(c)}]|0;0\rangle$ one finds $A^c(2) = -17$ as shown in Appendix F. These two relations suffice to determine the constants c_1 and c_3 in (140) such that

$$A^c(m) = \frac{1}{6}\left(m - 13m^3\right). \tag{141}$$

After evaluating the ghost contribution one can now define the complete Virasoro generators corresponding to both Polyakov and ghost action

$$L_m = L_m^{(\alpha)} + L_m^{(c)} - a\delta_m. \tag{142}$$

Here the L_0 operator was shifted by $-a$ such that the new constraint is $L_0 = 0$. This gives rise to an extra term in the anomaly of $2am$ such that the full anomaly is given by

$$A(m) = \frac{D}{12}\left(m^3 - m\right) + \frac{1}{6}\left(m - 13m^3\right) + 2am. \tag{143}$$

4.6 BRST Operator and Ghost Number

Besides the BRST operator Q there is a second conserved quantity in the system, the ghost number U. Both of these can be derived as integrals over conserved currents [14]. The BRST current is given by

$$J_-^B = 2c^- \left(T_{--}^{(\alpha)} + \frac{1}{2} T_{--}^{(c)} \right) \tag{144}$$

where J_+^B is obtained by replacing $-$ by $+$. $T_{--}^{(\alpha)}$ is given in (45) and $T_{--}^{(c)}$ in (134). The ghost number current is defined as

$$J_- = c^- b_{--} \tag{145}$$

where again J_+ is obtained via $- \leftrightarrow +$. Using the conservation of energy and momentum $\partial_+ T_{--} + \partial_- T_{+-} = 0$ which for the traceless case reduces to $\partial_+ T_{--} = 0$ and the equations of motion of b and c (126) one can show that these currents are indeed conserved,

$$\partial_+ J_-^B = \partial_+ J_- = 0. \tag{146}$$

The conserved charges corresponding to these currents are the BRST charge

$$Q = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left(J_+^B + J_-^B \right) \tag{147}$$

and the ghost number

$$U = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left(J_+ + J_- \right) \tag{148}$$

Focussing on the right-moving ghosts and oscillators, the BRST charge can be calculated as (see Appendix G)

$$Q = \sum_{-\infty}^{\infty} L_{-m}^{(\alpha)} c_m - \frac{1}{2} \sum_{-\infty}^{\infty} (m-n) : c_{-m} c_{-n} b_{m+n} : - a c_0. \tag{149}$$

Looking at the definition of $L_n^{(c)}$ this can be written as

$$Q = \sum_{-\infty}^{\infty} : \left(L_{-m}^{(\alpha)} + \frac{1}{2} L_{-m}^{(c)} - a \delta_m \right) c_m : . \tag{150}$$

Obtaining Q in this form is important as it allows one to check that Q is indeed nilpotent, which is absolutely crucial for the BRST mechanism. One can calculate

$$Q^2 = \frac{1}{2} \{Q, Q\} = \frac{1}{2} \sum_{-\infty}^{\infty} \left([L_m, L_n] - (m-n) L_{m+n} \right) c_{-m} c_{-n} \tag{151}$$

where L_n is given by (142). Clearly this is only 0 if the full Virasoro algebra is anomaly free. From (138) one can see that this is only the case for $D = 26$ and $a = 1$, therefore confirming the values obtained with the Old Covariant Quantisation.

Analogously to before one can integrate over the ghost number current to obtain the ghost number (see Appendix G),

$$U = \sum_{n=-\infty}^{\infty} : c_n b_{-n} : \tag{152}$$

where again this expression only includes right-moving ghosts. It is instructive to write out U in an explicitly normal ordered form

$$U = \frac{1}{2}\left(c_0 b_0 - b_0 c_0\right) + \sum_{n=1}^{\infty}\left(c_{-n} b_n - b_{-n} c_n\right) + a' \tag{153}$$

where the minus signs appear because c_n and b_{-n} anti-commute and a new normal ordering constant a' was introduced. Even though this is not shown here both c_0 and b_0 commute with the Hamiltonian, such that the ground state has a degeneracy. Let this degeneracy be described by two states $|\uparrow\rangle$ and $|\downarrow\rangle$ that are annihilated respectively by c_0 and b_0. Using the anti-commutation relations $c_0^2 = b_0^2 = 0$ and $\{c_0, b_0\} = 1$ the ground states must obey

$$c_0|\downarrow\rangle = |\uparrow\rangle \quad , \quad b_0|\uparrow\rangle = |\downarrow\rangle. \tag{154}$$

From (153) one can see that the ghost numbers of $|\uparrow\rangle$ and $|\downarrow\rangle$ obey $U_\uparrow = U_\downarrow + 1$. However, because of the normal ordering constant a', this relation does not directly fix the individual values. Nevertheless one can choose $a' = 0$ as this is the most symmetric choice giving $U_\uparrow = \frac{1}{2}$ and $U_\downarrow = -\frac{1}{2}$.
The reason why the ghost number was investigated so thoroughly here is that it is important for finding the physical states in the enlarged Hilbert space. As a physical state $|\psi\rangle$ is free of ghosts it should be annihilated by all ghost and anti-ghost annihilation operators,

$$c_n|\psi\rangle = b_n|\psi\rangle = 0, \quad n > 0 \tag{155}$$

One would also expect it to be in one of the two ghost ground states $|\uparrow\rangle$ or $|\downarrow\rangle$. At this point there is no obvious reason to choose one over the other. This however is not just a matter of convention as the ghost field c and anti-ghost field b do not enter the theory symmetrically. Choosing $|\downarrow\rangle$ a physical state would have ghost number $-\frac{1}{2}$ and be annihilated by b_0. For this choice something interesting happens. The condition of BRST invariance $Q|\chi\rangle = 0$ reduces to

$$0 = Q|\chi\rangle = \left(c_0\left(L_0^{(\alpha)} - 1\right) + \sum_{n>0} c_{-n} L_n^{(\alpha)}\right)|\psi\rangle \tag{156}$$

which can be seen from (149). These conditions are exactly the conditions on a physical state found in the Old Covariant Quantisation which therefore confirms that states with

ghost number $-\frac{1}{2}$ are indeed physical states.[3]
However, just like in the Old Covariant Quantisation one finds that certain states are equivalent to each other. More specifically adding $Q|\lambda\rangle$ to a physical state $|\psi\rangle$ will not change the overlap with any other physical state $|\phi\rangle$ as $\langle\phi|(|\psi\rangle + Q|\lambda\rangle) = \langle\phi|\psi\rangle$ as $|\phi\rangle$ is annihilated by Q. One can therefore identify $|\psi\rangle + Q|\chi\rangle$ with $|\psi\rangle$ such that

$$\mathcal{H}_{physical} = \Big\{|\psi\rangle \in \mathcal{H} \,:\, Q|\psi\rangle = 0\,,\, U|\psi\rangle = -\frac{1}{2}|\psi\rangle\Big\}/\Big\{|\psi\rangle \sim |\psi\rangle + Q|\chi\rangle\Big\}. \tag{157}$$

which means that in the BRST prescription physical states correspond to cohomology classes of ghost number $-\frac{1}{2}$.

[3]States with ghost number $\frac{1}{2}$ are annihilated by c_0 instead of b_0 and this means that BRST invariance does not reproduce all the physicality conditions.

5 Compactification and T-duality

This section will look at how the previous analysis of strings can be adapted to the case when they are moving in a compactified dimension. It will be shown that the change in boundary conditions of the string coordinates leads to the emergence of winding modes and quantisation of momentum. Those two are linked by a remarkable symmetry, T-duality, which describes that they can be interchanged if one inverts the radius of the compactified dimension. Gauging the string action in a curved background the Buscher rules are derived which state how the metric and background field transform under a T-dual transformation. Finally the analysis is generalised to compactification on a d-dimensional torus and selected elements of the group generating T-dual transformations are examined.

5.1 1-Dimensional Compactification

It was shown thoroughly in previous sections that string theory requires the existence of 26 space-time dimensions. As it seems impossible to have multiple time-like dimensions, the 22 extra dimensions are thought to be compactified spatial dimensions. Compactified means that they are curled up in space and periodic, for example the simplest one-dimensional compactification is a circle and the effect this has on the string spectrum will be analysed in the following.

Assume the string is moving in the background $\mathbf{R}^{1,24} \times \mathbf{S}^1$ where the compactified dimension is a circle of radius R and which will be relabelled as $Y = X^{25}$. This changes the string dynamics in two ways:

First, it modifies the boundary condition of the mode expansion along the circle. Rather than requiring $Y(\sigma + 2\pi) = Y(\sigma)$ it is sufficient to require

$$Y(\sigma + 2\pi) = Y(\sigma) + 2\pi w R \quad m \in \mathbf{Z}. \tag{158}$$

The integer w is the number of times the string wraps around $\mathbf{S}^1$ and is usually referred to as the winding number.

The second modification can be seen by demanding that the string wavefunction must be single valued along the circle,

$$\psi(X^i, Y + 2\pi R) = \psi(X^i, Y) \tag{159}$$

Noting that the wavefunction includes the factor $e^{ip\cdot X}$ this means that $e^{ip^Y 2\pi R} = 1$ such that the string momentum along the circle is quantised

$$p^Y = \frac{n}{R} \quad , \quad n \in \mathbf{Z}. \tag{160}$$

These two changes only affect the mode expansion for the string coordinates on the circle. The others simply remain (42). For Y the general mode expansion is

$$Y(\sigma,\tau) = y + \frac{\alpha' n}{R}\tau + wR\sigma + \text{oscillator modes} \tag{161}$$

which ensures that both (158) and (160) are obeyed. Once again Y can be split up into a left and right moving part, $Y(\sigma,\tau) = Y_L(\xi^+) + Y_R(\xi^-)$, where

$$\begin{aligned} Y_L\left(\xi^+\right) &= \frac{1}{2}y + \frac{1}{2}\alpha' p_L \xi^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{25} e^{-in\xi^+} \\ Y_R\left(\xi^-\right) &= \frac{1}{2}y + \frac{1}{2}\alpha' p_R \xi^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25} e^{-in\xi^-} \end{aligned} \tag{162}$$

and the left and right moving momenta were introduced as

$$p_L = \frac{n}{R} + \frac{wR}{\alpha'} \quad , \quad p_R = \frac{n}{R} - \frac{wR}{\alpha'}. \tag{163}$$

One can now determine how the mass spectrum looks to an observer in the 25 non-compact dimensions. The mass of the particle is given by

$$M^2 = -\sum_{\mu=0}^{24} p_\mu p^\mu \tag{164}$$

As before the conditions on L_0 and $\tilde{L}_0$ fix the mass. Starting from $L_0 - a = 0$ and using $a = 1$ one finds

$$\begin{aligned} L_0 - 1 &= \frac{1}{2}\alpha_0^2 + \sum_{m=1}^{\infty}\alpha_{-m}\cdot\alpha_m - 1 \\ &= \frac{\alpha'}{2}\sum_{\mu=0}^{24} p_\mu p^\mu + \frac{\alpha'}{2}p_R^2 + \sum_{m=1}^{\infty}\alpha_{-m}\cdot\alpha_m - 1 \\ &= -\frac{\alpha'}{2}M^2 + \frac{\alpha'}{2}p_R^2 + \sum_{m=1}^{\infty}\alpha_{-m}\cdot\alpha_m - 1 = 0. \end{aligned} \tag{165}$$

Therefore, defining

$$N \equiv \sum_{m=1}^{\infty}\alpha_{-m}\cdot\alpha_m \tag{166}$$

and rearranging one finds,

$$M = p_R^2 + \frac{4}{\alpha'}(N-1) \tag{167}$$

Repeating the same exercise for the left-moving modes one also finds

$$M = p_L^2 + \frac{4}{\alpha'}(\tilde{N} - 1) \tag{168}$$

where

$$\tilde{N} \equiv \sum_{m=1}^{\infty} \tilde{\alpha}_{-m} \cdot \tilde{\alpha}_m. \tag{169}$$

Adding and subtracting (167) and (168) gives

$$N - \tilde{N} = nw \tag{170}$$

and

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2). \tag{171}$$

Therefore both, the momentum along the circle as well as the winding modes, contribute to the mass of the string. If $n > 0$ the momentum gives a contribution of $\frac{n}{R}$ and for $w > 0$ the winding modes contribute with $2\pi wRT$, where $T = 1/2\pi\alpha'$ is the tension of the string [13].

It is useful to investigate the behaviour of the mass in the limit of a very large and very small radius R. As $R \to \infty$ the winding modes proportional to R/α' become very heavy and energetically unfavourable. The momentum modes, however, become very light and start to form a continuum as they would for a non-compact dimension. This was of course expected as the limit $R \to \infty$ is just a non-compact dimension. Also $p_L = p_R$ such that one recovers exactly the non-compactified mode expansions.

An interesting behaviour arises as $R \to 0$. This time the momentum states become infinitely heavy and decouple, and the winding modes start to form a continuum. Mysteriously the energy spectrum looks as if an extra uncompactified dimension has appeared!

5.2 T-duality

The emergence of an extra non-compact dimension stands in marked contrast to what would occur in quantum field theory and only occurs in string theory. It can be explained with the concept of T-duality. T-duality describes a remarkable property of (171): the mass spectrum remains invariant under the simultaneous exchanges

$$\boxed{n \longleftrightarrow w, \quad R \longleftrightarrow R' = \frac{\alpha'}{R}} \tag{172}$$

Therefore a string moving on a circle of radius R is exactly equivalent to a string moving on a circle with radius α'/R, but with the winding modes and momenta interchanged. Under the transformation (172)

$$p_L \longrightarrow p_L \quad \text{and} \quad p_R \longrightarrow -p_R. \tag{173}$$

Because of these transformations one therefore finds that

$$\partial_+ Y' = \partial_+ Y \tag{174}$$

$$\partial_- Y' = -\partial_- Y \tag{175}$$

This in turn gives

$$\partial_\tau Y' = \partial_\sigma Y \tag{176}$$

$$\partial_\sigma Y' = \partial_\tau Y \tag{177}$$

which can be written more compactly as

$$\partial_m Y' = \epsilon_{mn} \partial^n Y = \epsilon_m{}^n \partial_n Y \tag{178}$$

where

$$\epsilon_{mn} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{179}$$

From (178) one sees that Y and $\tilde{Y}$ seem to be naturally related. In fact one can show that the two are equivalent at the level of the action (see Appendix H), therefore confirming that they are just two descriptions of the same physics.

5.3 Buscher-Procedure

The natural generalisation of the previous part is to see how the string action transforms in a general curved background including an antisymmetric B-field. This will lead to a set of transformation rules for g_{ij} and B_{ij} first derived by Buscher [19]. Because of their great importance the derivations of those is included here. For convenience one can choose to work with dimensionless coordinates in the compactified dimension Y such that $Y \sim Y + 2\pi$ and the metric $G_{YY} = R^2$. In the conformal gauge the general action in the curved background is

$$\begin{aligned} S &= \frac{1}{\pi} \int d^2\xi (g+B)_{ij} \partial_+ X^i \partial_- X^j \\ &= \frac{1}{\pi} \int d^2\xi \left(G_{YY} \partial_+ Y \partial_- Y + E_{iY} \partial_+ X^i \partial_- Y + E_{Yi} \partial_+ Y \partial_- X^i + E_{ij} \partial_+ X^i \partial_- X^j \right) \end{aligned} \tag{180}$$

where $E = g + B$. This action has $U(1)$ gauge symmetry for Y. One can therefore make this action invariant under $Y \to Y + \alpha$ by introducing a gauge field A_m which transforms as $A_m \to A_m - \partial_m \alpha$. The covariant derivative $D_m Y = \partial_m Y + A_m$ is then invariant under transformation. Replacing partial derivatives by covariant derivatives and introducing the gauge fix $F_{mn} = \partial_m A_n - \partial_n A_m = 0$ using a Lagrange multiplier $\tilde{Y}$ the action in light-cone coordinates becomes [20]

$$\begin{aligned} S_{\text{gauged}} = \frac{1}{\pi} \int d^2\xi \big[& G_{YY}(\partial_+ Y + A_+)(\partial_- Y + A_-) + E_{iY}\partial_+ X^i(\partial_- Y + A_-) \\ & + E_{Yi}(\partial_+ Y + A_+)\partial_- X^i + E_{ij}\partial_+ X^i \partial_- X^j + \tilde{Y}(\partial_+ A_- - \partial_- A_+) \big] \end{aligned} \tag{181}$$

One can integrate the Lagrange multipliers by parts so that the gauge fields $A_\pm$ have algebraic equations of motion,

$$\text{From } A_- : \qquad \partial_+ X^i E_{iY} + (\partial_+ Y + A_+)G_{YY} - \partial_+ \tilde{Y} = 0 \tag{182}$$

$$\text{From } A_+ : \qquad \partial_+ X^i E_{Yi} + (\partial_- Y + A_-)G_{YY} + \partial_- \tilde{Y} = 0 \tag{183}$$

Rearranging and substituting these expressions for A_+ and A_- into (181) the action becomes

$$\begin{aligned} S_{\text{dual}} = \frac{1}{4\pi} \int d^2\xi \Big(& \frac{1}{G_{YY}} \partial_+ \tilde{Y} \partial_- \tilde{Y} - \frac{1}{G_{YY}} E_{iY} \partial_+ X^i \partial_- \tilde{Y} + \frac{1}{G_{YY}} E_{Yi} \partial_+ \tilde{Y} \partial_- X^i \\ & + \big(E_{ij} - \frac{E_{iY}E_{Yj}}{G_{YY}}\big) \partial_+ X^i \partial_- X^j \Big) \end{aligned} \tag{184}$$

But this action is exactly the same as (181) under the redefinitions

$$G_{YY} \to \frac{1}{G_{YY}}, \quad E_{Yi} \to \frac{E_{Yi}}{G_{YY}}, \quad E_{iY} \to -\frac{E_{iY}}{G_{YY}}, \quad E_{ij} \to E_{ij} - \frac{E_{iY}E_{Yj}}{G_{YY}}, \tag{185}$$

which are the famous Buscher rules [21].

5.4 d-Dimensional Compactification

Consider now the situation where one has a theory with d compactified dimensions Y^i of unit radius where $i = 1, 2, ..., d$. They are periodic and each have a winding number w^i associated with them such that

$$Y^i(\sigma + 2\pi) = Y^i(\sigma) + 2\pi w^i. \tag{186}$$

Their momentum along the direction of the compactified dimension is again quantised with the integers n_i. As always all states need to obey the physicality conditions of the L_n operators. For the purpose of this argument assume that all excited states are turned off such that all the oscillator modes vanish and the only non-trivial conditions are those for L_0 and $\tilde{L}_0$. These also have to be obeyed after any T-dual transformation and this allows one to find the group of allowed symmetry transformations.
Starting from the L_0 and $\tilde{L}_0$ conditions in the form (168) and (167) one can add and subtract those to find

$$p_L^2 - p_R^2 = 0 \tag{187}$$

and

$$p_R^2 + p_L^2 = 2M + \frac{4}{\alpha'}. \tag{188}$$

where N and $\tilde{N}$ were set to zero. Note that now $p_R^2 = p_R^i p_{Ri}$ where $i = 1, 2, ..., d$ runs over all compactified dimensions and similarly for p_L. As shown in Appendix I these two equations imply

$$n_i w^i = 0. \tag{189}$$

and

$$2M + \frac{4}{\alpha'} = \frac{2}{\alpha'^2}\big(g^{ik}(n_k + w^j B_{kj})(n_i + w^l B_{il}) + w^j w_j\big) \tag{190}$$

where g_{ij} is the metric and B_{ij} the antisymmetric background field. Defining

$$\mathbf{N} = \begin{pmatrix} w^i \\ n_i \end{pmatrix}. \tag{191}$$

both conditions can be expressed neatly in matrix form as

$$\mathbf{N}^T \eta \mathbf{N} = 0 \qquad \text{and} \qquad \mathbf{N}^T \mathbf{G} \mathbf{N} = \frac{\alpha'^2}{2}(2M + \frac{4}{\alpha'}) \tag{192}$$

where

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad , \quad \mathbf{G} = \begin{pmatrix} g_{jl} + g^{ik} B_{kj} B_{il} & g^{jk} B_{kl} \\ g^{jk} B_{kl} & g^{jl} \end{pmatrix} \tag{193}$$

The elements of these matrices are $d \times d$ matrices themselves. These conditions must always hold. Specifically they must also hold after any T-dual transformation $\mathbf{M}$ has acted on $\mathbf{N}$ to change the winding and momentum numbers. The r.h.s. of the both conditions in (192) are obviously invariant under a transformation $\mathbf{N} \rightarrow \mathbf{N}' = \mathbf{M}\mathbf{N}$. Invariance of the first condition implies

$$(\mathbf{M}\mathbf{N})^T \eta \mathbf{M}\mathbf{N} = \mathbf{N}^T \eta \mathbf{N} \tag{194}$$

such that

$$\boxed{\mathbf{M}^T \eta \mathbf{M} = \eta} \tag{195}$$

Similarly, the second condition implies

$$\mathbf{N}'^T \mathbf{G}' \mathbf{N}' = \mathbf{N}^T \mathbf{G} \mathbf{N} \tag{196}$$

and therefore

$$\boxed{\mathbf{G}' = (\mathbf{M}^T)^{-1} \mathbf{G} \mathbf{M}^{-1}} \tag{197}$$

The first boxed equation (195) defines the group of allowed symmetry operations $\mathbf{M}$ which is given by the orthogonal group $O(d, d, \mathbb{Z})$. The second boxed equation (197) gives the transformation of $\mathbf{G}$ under such a symmetry operation.

5.5 The $O(d,d,\mathbb{Z})$ Group

In order to understand the $O(d,d,\mathbb{Z})$ group better it is convenient to write its elements as

$$\mathbf{M} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \tag{198}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are $d \times d$ matrices. From the group defining property (195) one finds that these are related to each other through

$$\mathbf{a}^T\mathbf{c} + \mathbf{c}^T\mathbf{a} = 0, \quad \mathbf{b}^T\mathbf{d} + \mathbf{d}^T\mathbf{b} = 0, \quad \mathbf{a}^T\mathbf{d} + \mathbf{c}^T\mathbf{b} = \mathbb{1}. \tag{199}$$

Using those one can define three types of symmetry operators. Those are analysed for the simplest case when $g_{ij} = R^2\delta_{ij}$ and $B_{ij} = 0$:

1. Basis change $\mathbf{a} \in GL(d,\mathbb{Z})$
   By setting $\mathbf{b} = \mathbf{c} = 0$ one obtains the $O(d,d,\mathbb{Z})$ element

   $$\mathbf{M} = \begin{pmatrix} \mathbf{a} & 0 \\ 0 & (\mathbf{a}^T)^{-1} \end{pmatrix}. \tag{200}$$

   These elements correspond to basis changes of the compactified lattice. They manifest themselves because there is a mathematical redundancy in the description of the torus. Choose for example to look at

   $$\mathbf{a} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{201}$$

   This matrix just interchanges the x-axis and the y-axis and therefore changes the description of the torus but leaves it physically unchanged. Calculating $\mathbf{G}'$ explicitly one can confirm that this equals $\mathbf{G}$ as expected.

2. Integer shift of B-field
   Another type of symmetry operations can be obtained by letting $\mathbf{b} = 0$ and $\mathbf{a} = \mathbb{1}$. From (199) this automatically sets $\mathbf{d} = \mathbb{1}$ and requires $\mathbf{c}$ to be antisymmetric. Checking explicitly how $\mathbf{G}$ transforms under such an operation gives

   $$\begin{aligned} \mathbf{G}' = \left(\mathbf{M}^T\right)^{-1}\mathbf{G}\mathbf{M}^{-1} &= \begin{pmatrix} \mathbb{1} & -\mathbf{c}^T \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbf{g} & 0 \\ 0 & \mathbf{g}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -c & \mathbb{1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{1} & \mathbf{c} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbf{g} & 0 \\ -\mathbf{g}^{-1}\mathbf{c} & \mathbf{g}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{g} - \mathbf{c}\mathbf{g}^{-1}\mathbf{c} & \mathbf{c}\mathbf{g}^{-1} \\ -\mathbf{g}^{-1}\mathbf{c} & \mathbf{g}^{-1} \end{pmatrix} \end{aligned} \tag{202}$$

   But this is exactly equal to the original expression for $\mathbf{G}$ (193) with $\mathbf{B} = -\mathbf{c}$. This calculation therefore shows that one can add an antisymmetric matrix $\mathbf{c}$ composed of integers to $\mathbf{B}$ without changing the underlying physics. This is because any such shift changes the action by an integer multiple of 2π which leaves the path integral invariant [22].

3. Factorised duality D_i:
   The last group of symmetry operators making up the $O(d, d, \mathbb{Z})$ group are

$$\mathbf{M} = \begin{pmatrix} \mathbb{1} - \mathbf{e_i} & \mathbf{e_i} \\ \mathbf{e_i} & \mathbb{1} - \mathbf{e_i} \end{pmatrix} \tag{203}$$

   where $\mathbf{e_i}$ is 0 everywhere, except for the ii component which is 1. Even though these operators do not appear as obvious as the previous ones it can be checked that the matrices fulfil the conditions (199). Again looking at the two dimensional square torus case and using $\mathbf{e_0}$ then

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{204}$$

   Remembering the definition of the vector $\mathbf{N}$ (191) one sees that applying $\mathbf{M}$ onto it interchanges the winding modes into momentum modes and vice versa. Looking at the transformation metric one finds that

$$\mathbf{G}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R^2 & 0 & 0 & 0 \\ 0 & R^2 & 0 & 0 \\ 0 & 0 & R^{-2} & 0 \\ 0 & 0 & 0 & R^{-2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R^{-2} & 0 & 0 & 0 \\ 0 & R^2 & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^{-2} \end{pmatrix} \tag{205}$$

   This is equal to the original $\mathbf{G}$ except that the radius of the first dimension was inverted. This together with the interchanging of winding and momentum modes is therefore exactly the generalisation of the $R \to 1/R$ circle duality in the Y direction.

5.6 A Note on Superstrings and Mirror Symmetry

Interestingly enough superstrings turn out not to be invariant under T-duality. As hinted at in the introduction they rather map into each other under T-duality. Specifically string theory Type IIA on a circle of radius R maps onto Type IIB on a circle of radius α'/R. Similarly, the two heterotic string theories can be transformed into each other.

The idea that string theories can not distinguish between two different manifolds can also be generalised to more complicated cases and is known as mirror symmetry. This symmetry received its name because it holds if the hodge diamonds of two Calabi-Yau-manifolds $\mathbf{X}$ and $\mathbf{Y}$ in which the strings move are mirrors of each other. [13]

6 Conclusion

The last section on compactification and T-duality concludes this master thesis on bosonic string theory. After investigating the classical string in terms of its Lagrangian, equations of motion and symmetries, it was quantised in the flat gauge. This produced two problems relating to ordering ambiguities and the existence of ghost states. However, by imposing physicality conditions arising from the stress-energy-tensor as operator equations on the Hilbert space, ghost states could be removed from the theory in 26 dimensional spacetime and a value for the normal ordering constant was found. The same conditions were recovered by requiring nilpotency of the BRST operator in the BRST procedure. Here physical states naturally appeared as as the cohomology of this operator. Investigating the string dynamics in compactified dimensions an interesting symmetry became apparent, T-duality, which was analysed in terms of its symmetry operators that are elements of the $O(d, d, \mathbb{Z})$ group.

This is of course not the end of string theory, but rather just the beginning. In order to make contact with the real world one has to introduce supersymmetry. This removes the troublesome Tachyon that plagues bosonic string theory and allows the inclusion of fermions. Requiring the theory to be anomaly free one can show that it must live in 10-dimensions and studying dualities one can reach 11-dimensional M-theory. Here mirror symmetry and the beautiful ADS-CFT correspondence appear. However, one of the most interesting things about string theory is that some of the greatest ideas remain yet to be discovered. It is very much a work in progress. And only time will tell whether string theory will go down in history as the greatest idea that was not true or will be admired as the Theory of Everything.

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I also want to thank Professor Kellogg Stelle for assessing this project and the valuable feedback I have received for the literature review and my viva.

Just as important as the professors supervising and assessing this project were the students I was working with on a day to day basis. There was a day quite early on in the project when we were all gathered in lecture theatre 1, left with the task of deriving the existence of 26 space-time dimensions. We were standing in front of the board, loosing our selves in commutators and not making too much progress. This is when I realised how real research felt like. It is about being confused, and not knowing precisely what is going on because the moment you understand it, you move on. Very often this understanding came in working together with the three students with college ID 00941040, 01071261 and 01060899[4]. I would like to thank 00941040 for being a terrific project partner. Very smart, precise and always up for a joke, it has been a pleasure working with you. I also want to thank 01071261 for being a great friend and my fountain of all knowledge, at least physics related. Had you not made me pick Foundations of Quantum Mechanics in third year, I could not have done this amazing project. And of course I want to thank 01060899 who never shied away from endless calculations and livened up our conversations with strong opinions on literally everything.

[4]To ensure fair assessment college policy prohibits including the names of students in a master thesis.

Appendix A α_n^μ Commutation Relations

The string coordinates are given by

$$X^\mu = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{-in\xi^-} + \tilde{\alpha}_n^\mu e^{-in\xi^+} \right) \tag{206}$$

and the conjugate momenta by

$$\Pi_\mu = T\dot{X}_\mu = T\alpha' p_\mu + T\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left(\alpha_n^\mu e^{-in\xi^-} + \tilde{\alpha}_n^\mu e^{-in\xi^+} \right). \tag{207}$$

Their commutation relation is given by

$$[X^\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = i\delta(\sigma - \sigma')\,\delta_\nu^\mu. \tag{208}$$

Multiplying the expressions for X^μ and Π_μ by $e^{im\xi^-}$ and integrating over $d\xi^-$ from 0 to 2π one gets

$$\int_0^{2\pi} X^\mu e^{im\xi^-} d\xi^- = i\sqrt{2\pi^2\alpha'}\frac{1}{m}\alpha_m^\mu \tag{209}$$

and

$$\int_0^{2\pi} \Pi^\mu e^{im\xi^-} d\xi^- = \sqrt{\frac{1}{2\alpha'}}\alpha_m^\mu \tag{210}$$

where $T = \frac{1}{2\pi\alpha'}$. Therefore one can write

$$\alpha_m^\mu = \int_0^{2\pi} \left(\frac{-im}{\sqrt{8\pi^2\alpha'}} X^\mu + \sqrt{\frac{\alpha'}{2}}\Pi^\mu \right) e^{im\xi^-} d\xi^-. \tag{211}$$

Using this one can calculate the commutator at equal τ :

$$\begin{aligned}
[\alpha_m^\mu, \alpha_n^\nu] &= \int_0^{2\pi}\int_0^{2\pi} d\xi^- d\xi^{-\prime} \left(\frac{-im}{4\pi}\left[X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\right] - \frac{in}{4\pi}\left[\Pi^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\right] \right) e^{i(m\xi^- + n\xi^{-\prime})} \\
&= \int_0^{2\pi}\int_0^{2\pi} d\xi^- d\xi^{-\prime} \left(\frac{m}{4\pi}\eta^{\mu\nu}\delta(\sigma-\sigma') - \frac{n}{4\pi}\eta^{\mu\nu}\delta(\sigma-\sigma') \right) e^{i(m\xi^- + n\xi^{-\prime})} \\
&= \int_0^{2\pi} d\xi^- \left(\frac{m}{4\pi} - \frac{n}{4\pi} \right) \eta^{\mu\nu} e^{i(m+n)\xi^-} \\
&= \left(\frac{m}{4\pi} - \frac{n}{4\pi} \right) \eta^{\mu\nu} 2\pi\delta_{m+n} \\
&= m\eta^{\mu\nu}\delta_{m+n}
\end{aligned} \tag{212}$$

Appendix B Calculation of $[\alpha^\nu_m\,,\,L_n]$

When $n \neq 0$, L_n is given by (65), so that

$$\begin{aligned}
[\alpha^\nu_m\,,\,L_n] &= \frac{1}{2}\sum_p \left(\alpha^\nu_m \alpha^\mu_{n-p}\alpha^\sigma_p \eta_{\mu\sigma} - \alpha^\mu_{n-p}\alpha^\sigma_p \alpha^\nu_m \eta_{\mu\sigma}\right) \\
&= \frac{1}{2}\sum_p \left(\left(\alpha^\mu_{n-p}\alpha^\nu_m + m\eta^{\mu\nu}\delta_{m+n-p,0}\right)\alpha^\sigma_p \eta_{\mu\sigma} - \alpha^\mu_{n-p}\left(\alpha^\nu_m \alpha^\sigma_p + p\delta_{p+m,0}\eta^{\nu\sigma}\right)\eta_{\mu\sigma}\right) \\
&= \frac{1}{2}\sum_p \left(m\,\alpha^\sigma_p \delta_{m+n-p,0}\delta^\nu_\sigma - p\,\alpha^\mu_{n-p}\delta_{p+m,0}\delta^\nu_\mu\right) \\
&= m\,\alpha^\nu_{m+n}.
\end{aligned} \tag{213}$$

Checking L_0 given by (66) explicitly

$$\begin{aligned}
[\alpha^\nu_m\,,\,L_0] &= \left[\alpha^\nu_m\,,\,\frac{1}{2}\alpha^2_0 + \sum_{p=1}^\infty \alpha_{-p}\cdot\alpha_p\right] \\
&= \sum_{p=1}^\infty \left(\alpha^\nu_m \alpha^\mu_{-p}\alpha^\sigma_p \eta_{\mu\sigma} - \alpha^\mu_{-p}\alpha^\sigma_p \alpha^\nu_m \eta_{\mu\sigma}\right) \\
&= \sum_{p=1}^\infty \left(m\delta_{m-p,0}\alpha^\nu_p - p\,\delta_{p+m,0}\alpha^\nu_{-p}\right) \\
&= m\,\alpha^\nu_m
\end{aligned} \tag{214}$$

one finds that

$$[\alpha^\nu_m\,,\,L_n] = m\,\alpha^\nu_{m+n} \tag{215}$$

holds for all m and n.

Appendix C Check Physicality Conditions for First and Second Excited State

Here the physicality conditions

$$(L_n - a\delta_n)|\psi\rangle = 0 \quad , \quad n \geq 0 \tag{216}$$

will be applied on a general first and second excited state. For this choose to set $\alpha' = \frac{1}{2}$ such that

$$\alpha_0^\mu|0;p\rangle = \sqrt{\frac{\alpha'}{2}} p^\mu|0;p\rangle = \frac{p^\mu}{2}|0;p\rangle \tag{217}$$

Therefore

$$L_0|0;p\rangle = \frac{1}{2}\alpha_0^2|0;p\rangle = \frac{p^2}{8}|0;p\rangle. \tag{218}$$

C.1 First Excited State

Now evaluate those for a general first excited state $|1\rangle = \xi_\nu \alpha_{-1}^\nu|0;p\rangle$.

C.1.1 $(L_0 - a)|1\rangle = 0$

$$\begin{aligned}
(L_0 - a)\,|1\rangle &= \xi_\nu \left(\alpha_{-1}^\nu L_0 - [\alpha_{-1}^\nu, L_0] - a\alpha_{-1}^\nu\right)|0;p\rangle \\
&= \xi_\nu \left(\alpha_{-1}^\nu \frac{\alpha'}{4}p^2 + \alpha_{-1}^\nu - a\alpha_{-1}^\nu\right)|0;p\rangle \\
&= \left(\frac{\alpha'}{4}p^2 + 1 - a\right)|1\rangle \\
&= 0
\end{aligned} \tag{219}$$

Therefore giving the condition

$$\boxed{\frac{\alpha'}{4}p^2 = a - 1} \tag{220}$$

C.1.2 $L_1|1\rangle = 0$

$$\begin{aligned}
L_1|1\rangle &= L_1\xi_\nu\alpha_{-1}^\nu|0;p\rangle \\
&= \xi_\nu\alpha_0^\nu|0;p\rangle = \xi \cdot p\frac{1}{2}|0;p\rangle \\
&= 0
\end{aligned} \tag{221}$$

Therefore,

$$\boxed{\xi \cdot p = 0.} \tag{222}$$

For a first excited state those are the only conditions that have to be checked explicitly as all others are identically 0.

C.2 Second Excited State

A general second order excited state can be defined as

$$|2\rangle = \xi_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}|0;p\rangle + \tilde{\xi}_{\mu}\alpha^{\mu}_{-2}|0;p\rangle. \tag{223}$$

C.2.1 $(L_0 - a)|2\rangle = 0$

$$\begin{aligned}
L_0|2\rangle &= \Big[\xi_{\mu\nu}L_0\alpha^{\mu}_{-1}\alpha^{\nu}_{-1} + \tilde{\xi}_{\mu}L_0\alpha^{\mu}_{-2}\Big]|0;p\rangle \\
&= \Big[\xi_{\mu\nu}(\alpha^{\mu}_{-1}L_0 + \alpha^{\mu}_{-1})\alpha^{\nu}_{-1} + \tilde{\xi}_{\mu}(\alpha^{\mu}_{-2}L_0 + 2\alpha^{\mu}_{-2})\Big]|0;p\rangle \\
&= \Big[\xi_{\mu\nu}(\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}L_0 + \alpha^{\mu}_{-1}\alpha^{\nu}_{-1} + \alpha^{\mu}_{-1}\alpha^{\nu}_{-1}) + \tilde{\xi}_{\mu}(\alpha^{\mu}_{-2}L_0 + 2\alpha^{\mu}_{-2})\Big]|0;p\rangle \\
&= \Big[\xi_{\mu\nu}(\frac{p^2}{8} + 2)\alpha^{\mu}_{-1}\alpha^{\nu}_{-1} + \tilde{\xi}_{\mu}(\frac{p^2}{8} + 2)\alpha^{\mu}_{-2}\Big]|0;p\rangle \\
&= (\frac{p^2}{8} + 2)|2\rangle
\end{aligned} \tag{224}$$

Therefore,

$$(L_0 - a)|2\rangle = (\frac{p^2}{8} + 2 - a)|2\rangle = 0 \tag{225}$$

gives

$$\boxed{p^2 = -8} \tag{226}$$

as in the analysis of the first excited state it was shown that $a = 1$.

C.2.2 $L_1|2\rangle = 0$

$$\begin{aligned}
L_1|2\rangle &= \Big[\xi_{\mu\nu}L_1\alpha^{\mu}_{-1}\alpha^{\nu}_{-1} + \tilde{\xi}_{\mu}L_1\alpha^{\mu}_{-2}\Big]|0;p\rangle \\
&= \Big[\xi_{\mu\nu}(\alpha^{\mu}_{-1}L_1 + \alpha^{\mu}_{0})\alpha^{\nu}_{-1} + \tilde{\xi}_{\mu}(\alpha^{\mu}_{-2}L_1 + 2\alpha^{\mu}_{-1})\Big]|0;p\rangle \\
&= \Big[\xi_{\mu\nu}(\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}L_1 + \alpha^{\mu}_{-1}\alpha^{\nu}_{0} + \alpha^{\nu}_{-1}\alpha^{\mu}_{0}) + 2\tilde{\xi}_{\mu}\alpha^{\mu}_{-1}\Big]|0;p\rangle \\
&= \Big[\xi_{\mu\nu}(\frac{p^{\nu}}{2}\alpha^{\mu}_{-1} + \frac{p^{\mu}}{2}\alpha^{\nu}_{-1}) + 2\tilde{\xi}_{\mu}\alpha^{\mu}_{-1}\Big]|0;p\rangle
\end{aligned} \tag{227}$$

But from the definition of the $|2\rangle$ one can see that $\xi_{\mu\nu}$ must be symmetric, such that

$$L_1|2\rangle = \Big[\xi_{\mu\nu}p^{\nu}\alpha^{\mu}_{-1} + 2\tilde{\xi}_{\mu}\alpha^{\mu}_{-1}\Big]|0;p\rangle. \tag{228}$$

Therefore implying that

$$\boxed{\xi_{\mu\nu}p^{\nu} + 2\tilde{\xi}_{\mu} = 0} \tag{229}$$

C.2.3 $L_2|2\rangle = 0$

$$\begin{aligned}
L_2|2\rangle &= \left[\xi_{\mu\nu} L_2 \alpha^\mu_{-1}\alpha^\nu_{-1} + \tilde{\xi}_\mu L_2 \alpha^\mu_{-2}\right]|0;p\rangle \\
&= \left[\xi_{\mu\nu}(\alpha^\mu_{-1}L_2 + \alpha^\mu_1)\alpha^\nu_{-1} + \tilde{\xi}_\mu(\alpha^\mu_{-2}L_2 + 2\alpha^\mu_0)\right]|0;p\rangle \\
&= \left[\xi_{\mu\nu}(\alpha^\mu_{-1}\alpha^\nu_{-1}L_2 + \alpha^\mu_{-1}\alpha^\nu_1 + \alpha^\nu_{-1}\alpha^\mu_1 + \eta^{\mu\nu}) + 2\tilde{\xi}_\mu\alpha^\mu_0\right]|0;p\rangle \\
&= \left[\xi_{\mu\nu}\eta^{\mu\nu} + \tilde{\xi}_\mu p^\mu\right]|0;p\rangle
\end{aligned} \tag{230}$$

This gives

$$\boxed{\xi_{\mu\nu}\eta^{\mu\nu} + \tilde{\xi}_\mu p^\mu = 0} \tag{231}$$

Appendix D Calculation of the Stress-Energy Tensor

Using the formula for the world-sheet energy momentum tensor

$$T_{\alpha\beta} = -\frac{2\pi}{\sqrt{\gamma}}\frac{\delta S}{\delta\gamma^{\alpha\beta}} \tag{232}$$

one can find the contribution from the ghost action to the total energy-momentum tensor.

Varying the action with respect to the world-sheet metric one finds

$$\begin{aligned}\delta S_{ghost} = \frac{i}{2\pi}\int d^2\xi \Big[&\delta(\sqrt{\gamma})\gamma^{\alpha\beta}(\nabla_\alpha c^\mu)b_{\beta\mu} + \sqrt{\gamma}(\delta\gamma^{\alpha\beta})(\nabla_\alpha c^\mu)b_{\beta\mu} \\ &+ \sqrt{\gamma}\gamma^{\alpha\beta}\delta(\nabla_\alpha c^\mu)b_{\beta\mu} + \sqrt{\gamma}\gamma^{\alpha\beta}(\nabla_\alpha c^\mu)(\delta b_{\beta\mu})\Big]\end{aligned} \tag{233}$$

Using

$$\delta\sqrt{\gamma} = -\frac{1}{2}\sqrt{\gamma}\delta\gamma_{mn}\gamma^{mn} \tag{234}$$

and changing indices around the first term can be written as

$$\delta(\sqrt{\gamma})\gamma^{\alpha\beta}(\nabla_\alpha c^\mu)b_{\beta\mu} = -\frac{1}{2}\sqrt{\gamma}\delta(\gamma^{\alpha\beta})\gamma_{\alpha\beta}\gamma^{mn}(\nabla_m c^\mu)b_{n\mu} \tag{235}$$

The second term can be split up as

$$\sqrt{\gamma}(\delta\gamma^{\alpha\beta})(\nabla_\alpha c^\mu)b_{\beta\mu} = \frac{1}{2}\sqrt{\gamma}(\delta\gamma^{\alpha\beta})\left[(\nabla_\alpha c^\mu)b_{\beta\mu} + (\nabla_\beta c^\mu)b_{\alpha\mu}\right] \tag{236}$$

because $\gamma_{\alpha\beta}$ is symmetric.

The covariant derivative can be written in terms of Christoffel as

$$\nabla_\alpha c^\mu = \partial_\alpha c^\mu + \Gamma^\mu_{\alpha\nu}c^\nu \tag{237}$$

and the variation with respect to the Christoffel is

$$\delta\Gamma^\mu_{\alpha\nu} = \frac{1}{2}\gamma^{\mu\rho}\left(\nabla_\nu\delta\gamma_{\rho\alpha} + \nabla_\alpha\delta\gamma_{\rho\nu} - \nabla_\rho\delta\gamma_{\alpha\nu}\right) \tag{238}$$

Such that the third term becomes

$$\begin{aligned}\sqrt{\gamma}\gamma^{\alpha\beta}\delta(\nabla_\alpha c^\mu)b_{\beta\mu} &= \sqrt{\gamma}\gamma^{\alpha\beta}c^\nu\frac{1}{2}\gamma^{\mu\rho}\left(\nabla_\nu\delta\gamma_{\rho\alpha} + \nabla_\alpha\delta\gamma_{\rho\nu} - \nabla_\rho\delta\gamma_{\alpha\nu}\right)b_{\beta\mu} \\ &= \frac{1}{2}\sqrt{\gamma}c^\nu b^{\alpha\rho}\left(\nabla_\nu\delta\gamma_{\rho\alpha} + \nabla_\alpha\delta\gamma_{\rho\nu} - \nabla_\rho\delta\gamma_{\alpha\nu}\right) \\ &= \frac{1}{2}\sqrt{\gamma}c^\nu b^{\alpha\rho}\nabla_\nu\delta\gamma_{\rho\alpha} \\ &= -\frac{1}{2}\sqrt{\gamma}c^\nu b_{\alpha\beta}\nabla_\nu\delta\gamma^{\alpha\beta}\end{aligned} \tag{239}$$

In the last step the sign changed because $\delta(\gamma_{\rho\alpha}\gamma^{\alpha\beta}) = 0$ such that $\delta\gamma_{\rho\alpha} = -\gamma_{\rho\mu}\gamma_{\beta\alpha}\delta\gamma^{\mu\beta}$. The fourth term is 0. Pulling it all together and integrating the third term by parts one arrives at

$$\delta S_{ghost} = \frac{i}{2\pi}\int d^2\xi\sqrt{\gamma}\delta(\gamma^{\alpha\beta})\Big[-\frac{1}{2}\gamma_{\alpha\beta}\gamma^{mn}(\nabla_m c^\mu)b_{n\mu}+\frac{1}{2}\big[(\nabla_\alpha c^\mu)b_{\beta\mu}+(\nabla_\beta c^\mu)b_{\alpha\mu}\big]+\frac{1}{2}\nabla_\nu(c^\nu b_{\alpha\beta})\Big] \tag{240}$$

Because $b_{\alpha\beta}$ is traceless

$$\nabla_\nu(c^\nu b_{\alpha\beta})\delta(\gamma^{\alpha\beta}) = \nabla_\nu(c^\nu)b_{\alpha\beta}\delta(\gamma^{\alpha\beta}) + c^\nu(\nabla_\nu b_{\alpha\beta})\delta(\gamma^{\alpha\beta}) = c^\nu(\nabla_\nu b_{\alpha\beta})\delta(\gamma^{\alpha\beta}). \tag{241}$$

Using the definition of the stress-energy tensor (232), one finds the ghost contribution to the stress-energy tensor

$$T^{(c)}_{\alpha\beta} = -\frac{i}{2}\Big[(\nabla_\alpha c^\mu)b_{\beta\mu} + (\nabla_\beta c^\mu)b_{\alpha\mu} - \gamma_{\alpha\beta}\gamma^{mn}(\nabla_m c^\mu)b_{n\mu} + c^\nu\nabla_\nu(b_{\alpha\beta})\Big]. \tag{242}$$

In the conformal gauge the contribution to the energy-momentum tensor can be written in light-cone coordinates as

$$\begin{aligned} T^{(c)}_{++} &= -\frac{i}{2}\Big[(\partial_+ c^\mu)b_{+\mu} + (\partial_+ c^\mu)b_{+\mu} - \gamma_{++}\gamma^{mn}(\partial_m c^\mu)b_{n\mu} - c^\nu\partial_\nu(b_{++})\Big] \\ &= -\frac{i}{2}\Big[2(\partial_+ c^+)b_{++} + 2(\partial_+ c^-)b_{+-} + c^+\partial_+ b_{++} + c^-\partial_- b_{++}\Big]. \end{aligned} \tag{243}$$

Appendix E Ghost Contribution to Fourier Modes of the Stress-Energy Tensor

$L_n^{(c)}$ is defined as

$$L_n = T\int_0^{2\pi} d\sigma e^{-in\sigma}T_{--} = \frac{1}{\pi}\int_0^{2\pi} d\sigma e^{-in\sigma}T_{--} \tag{244}$$

in units where the string length $l = \frac{1}{\sqrt{\pi T}} = 1$. To calculate this first note that

$$\begin{aligned}
T_{--}^{(c)} &= -\frac{i}{2}\left[2\left(\partial_- c^-\right)b_{--} + c^-\partial_- b_{--}\right] \\
&= -\frac{i}{2}\Big[2\sum_{m=-\infty}^{+\infty}(-im)c_m e^{-im(\tau-\sigma)}\sum_{n=-\infty}^{+\infty} b_n e^{-in(\tau-\sigma)} \\
&+ \sum_{m=-\infty}^{+\infty} c_m e^{-im(\tau-\sigma)}\sum_{n=-\infty}^{+\infty}(-in)b_n e^{-in(\tau-\sigma)}\Big] \\
&= -\frac{1}{2}\sum_{n,m=-\infty}^{\infty} 2mc_m b_n e^{-i(m+n)(\tau-\sigma)}) - \frac{1}{2}\sum_{m,n=-\infty}^{+\infty} nc_m b_n e^{-i(m+n)(\tau-\sigma)} \\
&= -\frac{1}{2}\sum_{n,m=-\infty}^{\infty}(2m+n)c_m b_n e^{-i(m+n)(\tau-\sigma)} \\
&= -\frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{\alpha=-\infty}^{\infty}(2m+(\alpha-m))c_m b_{\alpha-m}e^{-i\alpha(\tau-\sigma)} \\
&= \frac{1}{2}\sum_{\alpha=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(-m-\alpha)c_m b_{\alpha-m}e^{-i\alpha(\tau-\sigma)}.
\end{aligned} \tag{245}$$

At $\tau = 0$,

$$\begin{aligned}
L_n^{(c)} &= \frac{1}{\pi}\int_0^{2\pi} d\sigma e^{-in\sigma}\frac{1}{2}\sum_{\alpha=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(-m-\alpha)c_m b_{\alpha-m}e^{i\alpha\sigma} \\
&= \frac{1}{2\pi}\sum_{\alpha=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(-m-\alpha)c_m b_{\alpha-m}\int_0^{2\pi} d\sigma e^{-in\sigma}e^{i\alpha\sigma}
\end{aligned} \tag{246}$$

But the integral

$$\int_0^{2\pi} d\sigma e^{i(\alpha-n)\sigma} = \left[\frac{1}{i(\alpha-n)}e^{i(\alpha-n)\sigma}\right]_0^{2\pi} = 0 \tag{247}$$

unless $n = \alpha$ in which case the integral is just 2π. Using this

$$
\begin{aligned}
L_n^{(c)} &= \frac{1}{2\pi} \sum_{\alpha=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-m-\alpha) c_m b_{\alpha-m} 2\pi \delta_{\alpha,n} \\
&= \sum_{m=-\infty}^{\infty} (-m-n) c_m b_{n-m} \\
&= \sum_{m=-\infty}^{\infty} (n-m) b_{n+m} c_{-m}
\end{aligned}
\tag{248}
$$

where in the last step $m \to -m$. An overall minus arises because c_m and b_n anti commute and as usual this expression has to be normal ordered.

Appendix F Matrix Elements for Ghost Commutators

F.1 Calculation of $A^c(1) = -2$

For the next two calculations note that of course $|0;0\rangle$ is a physical state and therefore contains no ghosts. Hence, operating with ghost and anti-ghost annihilation operators $c_{n>0}$ and $b_{n>0}$ on it gives 0,

$$c_n|0;0\rangle = b_n|0;0\rangle = 0 \quad \text{for} \quad n > 0. \tag{249}$$

Also from the definition of $L_n^{(c)}$ (135) note that $\langle 0;0|L_0^{(c)}|0;0\rangle = 0$. Therefore using the commutation relations (132) and (133),

$$\begin{aligned} A^c(1) &= \langle 0;0|[L_1^{(c)}, L_{-1}^{(c)}]|0;0\rangle = \langle 0;0|L_1^{(c)}L_{-1}^{(c)}|0;0\rangle \\ &= \langle 0;0|(-c_0b_1 - 2c_1b_0)(-b_{-1}c_0 - 2b_0c_{-1})|0;0\rangle \end{aligned} \tag{250}$$

But now terms that do not include b_1 and c_{-1} or vice versa vanish as one can just anti-commute the annihilation operators to the right and creation operators to the left. Hence,

$$\begin{aligned} A^c(1) &= \langle 0;0|2c_0b_1b_0c_{-1} + 2c_1b_0b_{-1}c_0|0;0\rangle \\ &= -2\langle 0;0|c_0b_0b_1c_{-1} + c_1b_{-1}b_0c_0|0;0\rangle \\ &= -2\langle 0;0|c_0b_0(\mathbb{1} - c_{-1}b_1) + (\mathbb{1} - b_{-1}c_1)b_0c_0|0;0\rangle \\ &= -2\langle 0;0|c_0b_0 + b_0c_0|0;0\rangle = -2 \end{aligned} \tag{251}$$

F.2 Calculation of $A^c(2) = -17$

Following exactly the same procedure as before,

$$\begin{aligned} A^c(2) &= \langle 0;0|[L_2^{(c)}, L_{-2}^{(c)}]|0;0\rangle = \langle 0;0|L_2^{(c)}L_{-2}^{(c)}|0;0\rangle \\ &= \langle 0;0|(2c_0b_2 + 3c_1b_1 + 4c_2b_0)(2b_{-2}c_0 + 3b_{-1}c_{-1} + 4b_0c_{-2})|0;0\rangle \\ &= \langle 0;0|8c_0b_2b_0c_{-2} + 9c_1b_1b_{-1}c_{-1} + 8c_2b_0b_{-2}c_0|0;0\rangle \\ &= -\langle 0;0|8c_0b_0 + 9c_1b_{-1} + 8b_0c_0|0;0\rangle \\ &= -8 - 9 = -17 \end{aligned} \tag{252}$$

Appendix G Calculation of BRST Charge and Ghost Number

G.1 BRST Charge

The BRST charge Q is defined as

$$Q=\frac{1}{2\pi}\int_0^{2\pi}d\sigma\left(J_+^B+J_-^B\right)=\frac{1}{2\pi}\int_0^{2\pi}d\sigma\left(2c^+(T_{++}^{(\alpha)}+\frac{1}{2}T_{++}^{(c)})+2c^-(T_{--}^{(\alpha)}+\frac{1}{2}T_{--}^{(c)})\right). \quad (253)$$

For this calculation the integral will only be performed over right-moving ghosts and oscillators as this shortens the algebra and one can just add an equivalent expression for the left-moving ghosts and oscillators at the end if needed. Using the form of $T_{--}^{(\alpha)}$ (48)

$$T_{--}^{(\alpha)}=\frac{1}{2}\left(\sum_{m=-\infty}L_m^{(\alpha)}e^{-im\sigma^-}-a\right) \quad (254)$$

where the a has to be included because L_0 was shifted and the result for $T_{--}^{(c)}$ that was derived in Appendix E,

$$T_{--}^{(c)}=\frac{1}{2}\sum_{\alpha=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(-m-\alpha)c_m b_{\alpha-m}e^{-i\alpha\sigma^-} \quad (255)$$

Q (at $\tau=0$) is given by

$$\begin{aligned}
Q&=\frac{1}{2\pi}\int_0^{2\pi}d\sigma\left(2\sum_{n=-\infty}c_ne^{in\sigma}\left(\frac{1}{2}\sum_{m=-\infty}L_m^{(\alpha)}e^{im\sigma}-\frac{a}{2}+\frac{1}{4}\sum_{\alpha=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(-m-\alpha)c_mb_{\alpha-m}e^{i\alpha\sigma}\right)\right)\\
&=\frac{1}{2\pi}\left(\sum_{n=-\infty}c_n\left(\sum_{m=-\infty}2\pi\delta_{n+m}L_m^{(\alpha)}-2\pi a\delta_n+\frac{1}{2}\sum_{\alpha=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}2\pi\delta_{n+\alpha}(-m-\alpha)c_mb_{\alpha-m}\right)\right)\\
&=\sum_{n=-\infty}c_nL_{-n}^{(\alpha)}+\frac{1}{2}\sum_{m,n=-\infty}^{\infty}(-m+n)c_nc_mb_{-n-m}-ac_0\\
&=\sum_{n=-\infty}c_nL_{-n}^{(\alpha)}-\frac{1}{2}\sum_{m,n=-\infty}^{\infty}(m-n)c_{-m}c_{-n}b_{m+n}-ac_0
\end{aligned} \quad (256)$$

As usual this expression needs to be normal ordered such that

$$Q=\sum_{n=-\infty}c_nL_{-n}^{(\alpha)}-\frac{1}{2}\sum_{m,n=-\infty}^{\infty}(m-n):c_{-m}c_{-n}b_{m+n}:-ac_0. \quad (257)$$

G.2 Ghost Number

The ghost number U is defined as

$$U = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \, (J_+ + J_-) = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left(c^+ b_{++} + c^- b_{--} \right) \tag{258}$$

As before the integral will be performed only over the right-moving ghosts as this simplifies the calculation and one can simply add the equivalent expression for the left-moving ghosts in the end. Inserting the mode expansions (128) and (130) (at $\tau = 0$)

$$\begin{aligned} U &= \frac{1}{2\pi} \int_0^{2\pi} d\sigma c^- b_{--} \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\sigma \sum_{m,n} c_n e^{in\sigma} b_m e^{im\sigma} \\ &= \frac{1}{2\pi} \sum_{m,n} (2\pi \delta_{m+n}) c_n b_m \\ &= \sum_{n=-\infty}^{\infty} : c_n b_{-n} : \end{aligned} \tag{259}$$

where in the last equation normal ordering was introduced by hand.

Appendix H T-dual Actions

Starting from the compactified part of the action one can make use of the $U(1)$ symmetry by making a gauge transformation. Fixing the gauge and integrating out the gauge field one can recover the T-dual counterpart of the original action, showing that both actions are really equivalent.
The original action is given by

$$S = \frac{1}{2\pi}\int d^2\xi(\partial Y)^2 R^2 \tag{260}$$

This action has $U(1)$ gauge symmetry. One can make this action invariant under $Y \to Y + \alpha$ by introducing a gauge field A_m which transforms as $A_m \to A_m - \partial_m\alpha$. The covariant derivative $D_m Y = \partial_m Y + A_m$ is now invariant under a transformation. Replacing partial derivatives by covariant derivatives and introducing the gauge fix $F_{mn} = \partial_m A_n - \partial_n A_m = 0$ using a Lagrange multiplier $\tilde{Y}$ the action becomes

$$S = \frac{1}{2\pi}\int d^2\xi(\partial_m Y + A_m)(\partial^m Y + A^m)R^2 + c\int d^2\xi\tilde{Y}\epsilon^{mn}F_{mn} \tag{261}$$

where c is a to be determined constant and ϵ is given by (179). Here, the Lagrange multiplier was suggestively called $\tilde{Y}$ as it will be shown below that this really is the the compactified dimension in the T-dual theory. Moving to light-cone coordinates and using integration by parts in the second step this evaluates to

$$\begin{aligned} S &= -\frac{1}{\pi}\int d^2\xi(\partial_+ Y + A_+)(\partial_- Y + A_-)R^2 - 4c\int d^2\xi\tilde{Y}(\partial_+ A_- - \partial_- A_+) \\ &= -\frac{1}{\pi}\int d^2\xi(\partial_+ Y + A_+)(\partial_- Y + A_-)R^2 + 4c\int d^2\xi(A_-\partial_+\tilde{Y} - A_+\partial_-\tilde{Y}) \end{aligned} \tag{262}$$

In this form A_+ and A_- are just auxiliary fields with algebraic equations of motion,

$$A_+ \qquad \to \qquad -\frac{1}{\pi}(\partial_- Y + A_-)R^2 - 4c\partial_-\tilde{Y} = 0 \tag{263}$$

$$A_- \qquad \to \qquad -\frac{1}{\pi}(\partial_+ Y + A_+)R^2 + 4c\partial_+\tilde{Y} = 0 \tag{264}$$

Rearranging those expression and substituting them in for A_+ and A_- the action becomes

$$\begin{aligned} S = &-\frac{1}{\pi}\int d^2\xi(\frac{4\pi c}{R^2}\partial_+\tilde{Y})(-\frac{4\pi c}{R^2}\partial_-\tilde{Y})R^2 \\ &+4c\int d^2\xi\Big((-\frac{4\pi c}{R^2}\partial_-\tilde{Y} - \partial_- Y)\partial_+\tilde{Y} - (\frac{4\pi c}{R^2}\partial_+\tilde{Y} - \partial_+ Y)\partial_-\tilde{Y}\Big) \end{aligned} \tag{265}$$

After recognising that

$$\int d^2\xi(-\partial_- Y\partial_+\tilde{Y} + \partial_+ Y\partial_-\tilde{Y}) = 0 \tag{266}$$

using integration by parts and reorganising terms the final expression for S is given by

$$S = -16\pi c^2 \int d^2\xi \partial_+\tilde{Y}\partial_-\tilde{Y}\frac{1}{R^2} = 8\pi c^2 \int d^2\xi(\partial\tilde{Y})^2\frac{1}{R^2} \tag{267}$$

Comparing this action to (260) one finds that they are identical under the T-dual transformations $\tilde{Y} = Y$ and $\tilde{R} = 1/R$ if $c = 1/4\pi$.

Appendix I D-Dimensional Compactification Conditions

Starting from

$$p_L^2 - p_R^2 = 0 \tag{268}$$

and

$$p_R^2 + p_L^2 = 2M + \frac{4}{\alpha'}. \tag{269}$$

one has to find expressions for the left and right moving momenta. The best way to do this is to return to the level of the action. Analogues to (180) the contribution of the compactified dimensions to the action is

$$S = \frac{1}{\pi}\int d^2\xi (g_{ij} + B_{ij})\partial_+ Y^i \partial_- Y^j = \frac{1}{4\pi}\int d^2\xi (g_{ij} + B_{ij})(\partial_\tau + \partial_\sigma)Y^i(\partial_\tau - \partial_\sigma)Y^j \tag{270}$$

From this equation one can calculate the canonical momentum $P_i = \frac{\partial \mathcal{L}}{\partial(\partial_\tau Y^i)}$ which is related to the center of mass momentum of the string by $p_i = 2\pi P_i$. Therefore,

$$\begin{aligned} p_k = 2\pi \frac{\partial \mathcal{L}}{\partial(\partial_\tau Y^k)} &= \frac{1}{2}(\partial_\tau - \partial_\sigma)Y^j(g_{kj} + B_{kj}) + \frac{1}{2}(\partial_\tau + \partial_\sigma)Y^i(g_{ik} + B_{ik}) \\ &= \dot{Y}^j g_{kj} - Y'^j B_{kj} \end{aligned}. \tag{271}$$

But the conjugate momenta are quantised $p_k = n_k$. Using (186) one can deduce that Y^i is of the form $Y^i(\sigma) = w^i\sigma + c$ where c is a constant such that $Y'^i = w^i$. Inserting those two relations into (271) and rearranging one finds that

$$\dot{Y}^i = g^{ik}(n_k + w^j B_{kj}). \tag{272}$$

Now this expression can be used to find the left and right moving momenta p_L and p_R. From (162)

$$p_L^i = \frac{2}{\alpha'}\partial_+ Y_L^i = \frac{2}{\alpha'}\partial_+ Y^i = \frac{1}{\alpha'}(\dot{Y}^i + Y'^i) \tag{273}$$

and likewise

$$p_R^i = \frac{1}{\alpha'}(\dot{Y}^i - Y'^i). \tag{274}$$

Therefore,

$$p_L^2 - p_R^2 = (p_L^i + p_R^i)(p_{Li} - p_{Ri}) = \frac{4}{\alpha'^2}\dot{Y}^i Y_i' = \frac{4}{\alpha'^2} g^{ik}(n_k + w^j B_{kj})w_i = \frac{4}{\alpha'^2} n_i w^i \tag{275}$$

because B_{ij} is antisymmetric. This implies

$$\boxed{n_i w^i = 0}. \tag{276}$$

In contrast

$$p_R^2 + p_L^2 = \frac{2}{\alpha'^2}(\dot{Y}^2 + Y'^2) = \frac{2}{\alpha'^2}\big(g^{ik}(n_k + w^j B_{kj})(n_i + w^l B_{il}) + w^j w_j\big) \tag{277}$$

such that the second condition is

$$\boxed{2M + \frac{4}{\alpha'} = \frac{2}{\alpha'^2}\big(g^{ik}(n_k + w^j B_{kj})(n_i + w^l B_{il}) + w^j w_j\big)}. \tag{278}$$

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